Summary 13: Functions from R^2 to *R* and Limits

Function: The function f associates to every point (x, y) in a set $D \subset R^2$ a number $z = f(x, y)$. The graph of such a function is a *surface* in R^3 and is best drawn using Mathematica's `Plot3D` function.

Level curves: The level curves for a function $f(x, y)$ are curves of constant height $f(x, y) = c$. They are best drawn using Mathematica's `ContourPlot` function, and they resemble a topographic map.

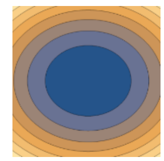
Limit: $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ means that given any $\varepsilon > 0$ there exists a number $\delta > 0$ such that whenever $||(x, y) - (x_0, y_0)|| < \delta$ we have that $|f(x, y) - L| < \epsilon$

How to find limits: First try the obvious: if substitution works, that is the answer. If you get 0/0 you try to find different approach paths such that you get different answers. Typically try:

- $x = 0, y \to 0$ (along y-axis)
- $y = 0, x \to 0$ (along x-axis)
- $x = y, x \to 0$ (along major diagonal)
- $y = x^2, x \to 0$ (along parabola)
- $x = y^2, y \to 0$ (along parabola)

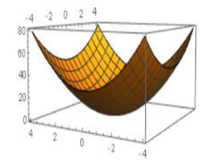
If *any two* answers *differ* from one another the *limit does not exists*. If all agree, perhaps the limit might exist (although not necessarily so). Try to prove it using the definition above or try other approach paths.

Plot the surface $z = 2x^2 + 3y^2$ and draw its level curves.



We could use Mathematica, but let's do it manually: contour plots $c = 2x^2 + 3x^2$ for different constants c.

Each such curve is an ellipse, so that the contour plot is on the left, while the actual surface is a paraboloid:



Find $\mathop{lim}_{(x,y)\to(0,0)} \frac{3xy^2}{x^3+4y^2}$

1. Subst. $x = 0, y = 0$: $\frac{0}{0}$ (more work)
2. $x = 0, y \to 0$: $\lim_{y\to 0} \frac{0}{4y^2} = 0$
3. $y = 0, x \to 0$: $\lim_{x\to 0} \frac{0}{x^2} = 0$
   Same limits, tells us nothing
4. $x = y, x \to 0$: $\lim_{x\to 0} \frac{3x^3}{x^3+4^2} = \frac{3}{5}$
   Different, so original limit d.n.e.