Summary 8: Lines, Planes, and Distances

Lines Parametric equation of a line though point P with *directional vector* $\vec{v}$ is: $l(t) = P_0 + t\,\vec{v} = < p_1, p_2, p_3 > +t < v_1, v_2, v_3 > =$ $= < p_1 + tv_1, p_2 + tv_2, p_3 + t\,v_3 >$	$l(t) =< 1,2,3 > +t\ < 3,0,1 >=$ $= < 1 + 3t, 2,3 + t >$
Planes The equation of a plane with normal vector $\vec{n} =< a, b, c >$ through the point $P(x_0, y_0, z_0)$ is: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or equivalently $ax + by + cz + d = 0$	$2(x + 1) - 3(y - 1) + (z - 3) = 0$ or equivalently $2x + 2 - 3y + 3 + z - 3 = 0$ or $2x - 3y + z + 2 = 0$

Distances

Distance between two points $P(x_0, y_0, z_0)$ *and* $Q(x_1, y_1, z_1)$: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$ *Distance between point* $P(x_0, y_0, z_0)$ *and plane* $ax + by + cz + d = 0$: Find any point Q on the plane. Then $d = \left\lVert proj_{\vec{n}}(\overrightarrow{PQ}) \right\rVert = \frac{\lvert \vec{n}\cdot\overrightarrow{PQ} \rvert}{\lVert \vec{n} \rVert}$ or $d = \frac{\lvert a\,x_0 + b\,y_0 + c\,z_0 + d \rvert}{\sqrt{a^2+b^2+c^2}}$ *Distance between line and plane:* First, check if they are parallel. If not, distance is zero. Otherwise pick any point on the line and use formula above *Distance between two planes:* First, check if they are parallel. If not, distance is zero. Otherwise pick any point on the second plane and use formula above *Distance between point Q and line:* If 3D problem: $d = \frac{\lVert \vec{v} \times \overrightarrow{PQ} \rVert}{\lVert \vec{v} \rVert}$. If 2D problem: $d = \frac{\lvert ax_0 + by_0 + d \rvert}{\sqrt{a^2+b^2}}$ *Distance between two lines* $l_1(t) = P + t\vec{v}_1$ *and* $l_2(t) = Q + t\vec{v}_2$: $d = \left\lVert proj_{v_1 \times v_2}(PQ) \right\rVert = \frac{\lvert (v_1 \times v_2) \cdot \overrightarrow{PQ} \rvert}{\lVert v_1 \times v_2 \rVert}$	*Distance between* $P(1,2,3)$ *and* $3x + 4y - 5z = 6$*:* Take $Q(2,0,0)$. Then: $d = \frac{<3,4,-5>\cdot<1,-2,-3>}{\sqrt{3^2+4^2+(-5)^2}} = \frac{10}{\sqrt{50}}$ or $d = \frac{\lvert 3*1+4*2-5*3-6 \rvert}{\sqrt{50}} = \frac{10}{\sqrt{50}}$ *Distance between* $x + y - 2z = 1$ *and* $l(t) =< 0,0,3 > +t\ < 4,2,3 >$*:* We have $< 1,1,-2 >\cdot< 4,2,3 >= 0$ so they are parallel. Pick $P(0,0,3)$ on the line and $Q(1,0,0)$ on the plane. Then: $d = \frac{<1,1,-2>\cdot<1,0,-3>}{\sqrt{1^2+1^2+(-2)^2}} = \frac{7}{\sqrt{6}}$ or $d = \frac{\lvert 0+0-6-1 \rvert}{\sqrt{6}} = \frac{7}{\sqrt{6}}$ *Distance between planes* $x + y - z = 1$ *and* $3x - 3y + 3z\ = 42$*:* planes are not parallel so distance is zero. *No examples necessary, these formulas are rarely used*

Note: the “take-away” formula you *really* need to know is that of the distance between a point and a plane or, consequently, between a line and a (parallel) plane or between two (parallel) planes.