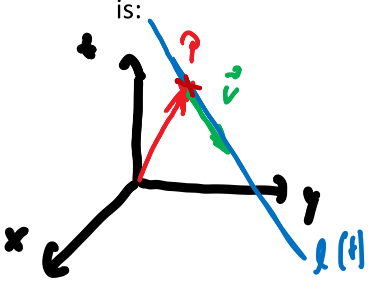
Distances in R^3

Last time we figured out the (parametric) equation of a line and the (scalar) equation of a plane:

Definition: The equation of a **line** through point $P(x_0, y_0, z_0)$ with directional vector $\vec{v} = < v_1, v_2, v_3 >$ is:

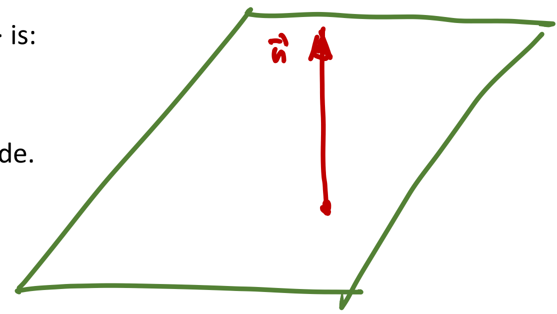


$$l(t) = P + t\vec{v} = < x_0 + t\, v_1, y_0 + t\, v_2, z_0 + t\, v_3 >$$

Definition: The equation of a **plane** with normal vector $\vec{n} = < a.b.c >$ is:

$$ax + by + cz + d = 0$$

where the number *d* depends on the point that the plane should include.

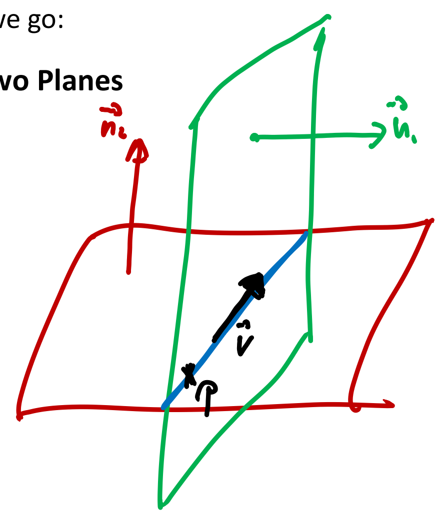


We discussed how to:

- Find out if a point is on a line *(set them equal, see if you can find a t that works for all 3 equations)*
- Find out if a point is on a plane *(plug in the x,y,z values of the point into the equation of the plane)*
- Find the intersection of a line and a plane *(plug in components of line into plane, solve for t)*
- Find the intersection of two lines *(try to solve three equations with two unknowns if possible)*
- Find the intersection of two planes *(we actually did not do this yet :)*

So, turns out we did *not* solve the last problem yet, so here we go:

Intersection of Two Planes



If two planes are parallel (i.e. their normal vectors are parallel), then they don't intersect. Otherwise they intersect in a line, so we need to find the point *P* and the directional vector $\vec{v}$ of that line. The vector $\vec{v}$ is part of the first plane, so it is perpendicular to its normal vector. But $\vec{v}$ is also part of the second plane, so it is also perpendicular to that normal vector. Hence, $\vec{v}$ is perpendicular to both normal vectors, which means that $\vec{v}$ is (parallel to) the cross product of both normal vectors: $\vec{v} = \vec{n}_1 \times \vec{n}_2$.

That leaves us to find any point *P(x, y, z)* on that line. The line, somewhere, will cross the xy-plane, so we can assume that its z-coordinate is zero. So, we have a point *P(x, y, 0)* that is part of both planes. That means that it has to satisfy two equations in two unknowns (since we could assume z = 0). We can solve that system and we have our point! Note that we could have set y = 0 and worked out the missing x and z coordinates, or x = 0 and worked out *y* and *z*. We would get different points, of course, but the resulting lines would be the same.

Example: Find the intersection of the planes $x - y + 2z = 3$ and $x + 2y - 3z = 0$

The planes have normal vectors $n_1 =< 1, -1, 2 >$ and $n_2 =< 1, 2, -3 >$. They are not parallel, so the planes are not parallel, either. Thus, there is a line $l(t) = P + \vec{v}t$ of intersection. As discussed above:

$$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 1 & 2 & -3 \end{vmatrix} =< 3 - 4, -(-3 - 2), 2 + 1 >=< -1,5,3 >$$

To find a point on that line, we may assume that z = 0. Thus, we get the system of equation

$$\begin{aligned} x - y &= 3 \\ x + 2y &= 0 \end{aligned}$$

Multiply the first equation by 2 and add that to the second one gives $3x = 6$ or $x = 2$. But then $y = -1$ so that the point is $P(2, -1, 0)$. Putting this point together with the vector $\vec{v}$ we worked out before gives the line of intersection $l(t) = (2, -1, 0) + t < -1,5,3 >=< 2 - t, -1 + 5t, 3t >$.

We can actually (and pretty easily) check answer. We're saying that the intersection of the planes $x - y + 2z = 3$ and $x + 2y - 3z = 0$ is the line $< 2 - t, -1 + 5t, 3t >$, and we know how to check if a line is on the plane: plug in $x = 2 - t, y = -1 + 5t, z = 3t$ into the first planes gives:

$$x - y + 2z = (2 - t) - (-1 + 5t) + 2(3t) \equiv 3$$

for all t, and plugging the same coordinates into the second plane again gives a true identity for all t:

$$x + 2y - 3z = (2 - t) + 2(-1 + 5t) - 3(3t) \equiv 0$$

Now that we have gotten everything possible out of the "intersection" questions, we will move on to the next (and last) topic in this chapter.

Distances between Points, Lines, and Planes

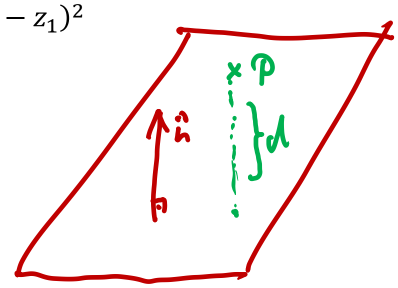
Next, we will determine distances. In particular, we want to compute:

- Distance between two points
- Distance between a point and a plane
- Distance between a point and a line
- Distance between a line and a plane
- Distance between two lines
- Distance between two planes

The first question is obvious and we have covered it quite some time ago: if $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

$$d(P, Q) = \left|\left|\overrightarrow{PQ}\right|\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

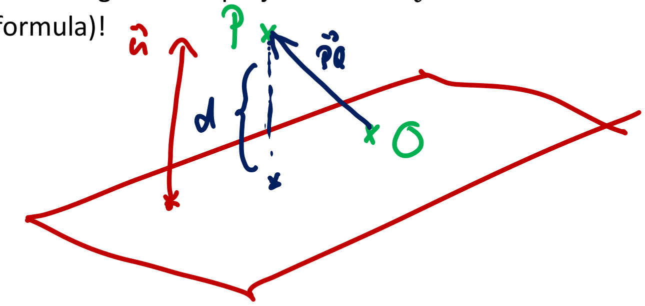
That was easy. Next, let's look at a plane and a point (not on the plane)



To find the distance between point P and the plane we will use the *projection formula*. In other words:

- Find an arbitrary point Q on the plane
- Find the vector $\overrightarrow{PQ}$

Then the distance *d* is the length of the projection on $\overrightarrow{PQ}$ onto the normal vector $\vec{n}$ (that's a rather clever way of using the projection formula)!



In other words:

$$d = \left\| proj_{\vec{n}}(\overrightarrow{PQ}) \right\| = \frac{\left[\vec{n} \cdot \overrightarrow{PQ}\right]}{\|\vec{n}\|}$$

To see a video that explains this again, check out https://www.youtube.com/watch?v=HW3LYLLc60I

Example: Find the distance between plane $x + 2y + 3z = 4$ and the point $P(3,2,1)$

We need any point Q on the plane. Setting *y = 0* and *z = 0* gives the point *Q(4, 0, 0)* on the plane. Then we find the vector $PQ = (4,0,0) - (3,2,1) =< 1,-2,-1 >$. Then the distance we want is the length of the projection of $PQ =< 1,-2,-1 >$ onto the normal vector of the plane $n =< 1,2,3 >$:

$$d = \frac{|< 1,2,3 > \cdot < 1,-2,-1 >|}{\|< 1,2,3 >\|} = \frac{|1-4-3|}{\sqrt{1+2^2+3^2}} = \frac{6}{\sqrt{14}}$$

We have to believe that the answer is correct, because it is difficult to verify the answer geometrically by drawing pictures in 3D. But if we take *special* planes and points, then we can easily see the answer and then use it to verify our formula. For example, find the distance between the xy-plane and the point P(4,3,5). It is easy to see that the distance here is 5, but what does our formula say? The equation of the xy plane is, simply, z = 0, which has the normal vector v = <0, 0, 1>. An arbitrary point on the xy-plane is Q(2, 3, 0) (or really any point whose z-coordinate is 0. Thus, according to our formula, the distance is:

$$d = \frac{|< 0,0,1 > \cdot PQ}{|| < 0,0,1 > ||} = \frac{|< 0,0,1 >< -2,0,-5 >|}{1} = 5$$

as it is supposed to. So, we verified our formula for this special case.

Here is another example: Use the above formula to verify that the distance between the point P(3,1,-2) and the plane x +4y + 3z = 1 is zero. Then explain this answer geometrically. *Hint: what is special about the point P(3,1,-2) in relation to the plane x + 4y + 3z = 1?*

We can actually get an easy-to-use formula from the above equation: take a point $P(x_0, y_0, z_0)$ and a plane $ax + by + cz + d = 0$. Pick some point $Q(x_1, y_1, z_1)$ on the plane (which of course means $ax_1 + by_1 +$

$cz_1 = -d$). Then the vector $\overrightarrow{PQ} = < x_1 - x_0, y_1 - y_0, z_1 - z_0 >$ and according to our formula we have the distance as

$$d = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\vec{n}||} = \frac{|a\,(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

But we already determined that $ax_1 + by_1 + cz_1 = -d$ so that:

> **Theorem:** The formula for the distance between plane $ax + by + cz + d = 0$ and the point $P(x_0, y_0, z_0)$ is:
>
> $$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example: Use this formula to find the distance between plane $x + 2y + 3z = 4$ and the point $P(3,2,1)$:

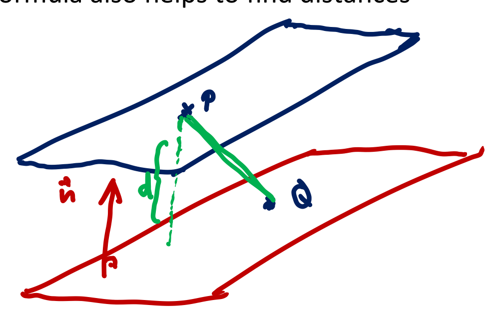
$$d = \frac{3 + 2 * 2 + 3 * 1 - 4}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{6}{\sqrt{14}}$$

That answer matches the previous answer, as it should - always good! As it turns out, we can use this formula to solve a 2D problem:

> **Theorem:** The distance between a line $ax + by + c = 0$ in R^2 and a point $P(x_0, y_0)$ in R^2 is:
>
> $$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

This is simply the previous formula reduced by one dimension. The above formula also helps to find distances between two planes and between a line and a plane.

Find the distance between two planes: first, determine if the planes are parallel. If not, their distance is zero. Why? Otherwise, find any point $P(x_0, y_0, z_0)$ on the *second* plane. Because the planes are parallel, the distance between the two planes is then given by the distance of that point to the *first* plane, for which the above formula applies.



Similarly, you can make up a recipe to find the distance between a line and a plane – do it.

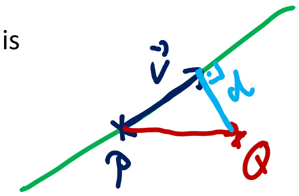
Examples: Here are a few examples. Remember to first check if the objects are parallel or not.

- Distance between $x - y + 2z = 1$ and $-2x + 2y - 2z = 4$
- Distance between $2x + 3y - z = 3$ and $4x + 6y - 2z = 5$
- Distance between $\mathrm{x + y + 2z = 0}$ and $< 1 - 2t, 3 + 4t, 3 - t >$
- Distance between $2x + 3y - z = 0$ and $< 1 + 3t, 2 + 2t, 3 + t >$

That leaves us with figuring out the distance between a point and a line in 3D and the distance between two lines. The formulas/procedures are getting rather complicated and are include here for completeness only, and without proof. First things first:

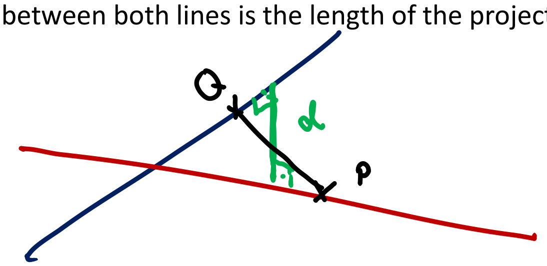
Theorem: The distance between point $Q(x_0, y_0, z_0)$ and the line $l(t) = P + vt$ is

$$\frac{||\overrightarrow{PQ} \times \vec{v}||}{||\vec{v}||}$$



Finally, consider two lines in R^3. If they intersect, their distance was clearly zero. If they don't intersect, use the following procedure to find their distance.

Theorem: To find the distance between two non-intersecting lines, compute the cross product of the two directional vectors and find a point P one the first line as well as a point Q on the second line. Then the distance between both lines is the length of the projection of the vector $\overrightarrow{PQ}$ onto the cross product of $v_1 \times v_2$.



Example: Find the distance between the point $Q(1,0,3)$ and the line $l(t) =< -1 - t, 3t, 2 >$ as well as the distance between that line and the line $l_2(t) =< 3 + 4t, 2 + 3t, 1 + t >$.

The line $l(t) =< -1 - t, 3t, 2 >$ goes through point $P(-1,0,2)$ with direction $v =< -1,3,0 >$ so that according to the above formula we have

$$d = \frac{||PQ \ \times v||}{||v||} = \frac{|| < 2,0,1 > \times < -1,3,0 > ||}{|| < -1,3,0 > ||} = \frac{|| < -3, -1,6 > ||}{|| < -1,3,0 > ||} = \frac{\sqrt{46}}{\sqrt{10}}$$

To find the distance between the two lines, we need $v_1 \times v_2 =< -1,3,0 > \times < 4,3,1 >=< 3,1, -15 >$. We take $P(-1,3,2)$ on line 1 and $Q(3,2,1)$ on line 2 so that, according to the recipe above:

$$d = ||proj_{<3,1,-15>}(< 4, -1, -1 >) = \frac{|< 3,1, -15 > \cdot < 4, -1, -1 >|}{|| < 3,1, -15 > ||} = \frac{12 - 1 + 15}{\sqrt{235}} = \frac{26}{\sqrt{235}}$$

That's it, we are done with distances, and in fact done with the entire chapter 11. Recall that we covered so far in this Calc 3 class:

- we started with points, spheres, and sheets in R^3
- vectors: add, subtract, multiply by scalar, length
- dot product: angles, perpendicular vector, projection
- cross Product: perp. to a and b
- lines: vector equation, parallel, perpendicular
- planes: scalar equation with normal vector
- intersections: point on line, point on plane, line and line, line and plane, plane and plane
- distances: points, points and lines, points and planes, lines and lines, lines and planes, plane and plane