Summary 6: Lines, Planes, and Intersections

Lines Parametric equation of a line though point P with *directional vector* $\vec{v}$ is: $l(t) = P_0 + t\,\vec{v} = < p_1, p_2, p_3 > + t < v_1, v_2, v_3 > =$ $= < p_1 + tv_1, p_2 + tv_2, p_3 + t\,v_3 >$	*Eq. of line through* $P(1,3,-2)$ *and* $Q(4,2,1)$*:* vector $\vec{v} = \overrightarrow{PQ} = Q - P = < 3,-1,3 >$ so that $l(t) = = (1,3,-2) + t < 3,-1,3 > = =< 1+3t, 3-t, -2+3t >$
Planes A *normal* vector to a plane is perpendicular to every vector in the plane. The equation of a plane with normal vector $\vec{n} = < a,b,c >$ through the point $P(x_0, y_0, z_0)$ is: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or equivalently $ax + by + cz + d = 0$	*Normal vector of plane* $2x - y + z = 9$*?* $\vec{n} = < 2,-1,1 >$ *Equation of plane through* $P(1,2,0)$*,* $Q(2,1,1)$*, and* $R(0,1,0)$*?* Normal vector $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = < 1,-1,1 > \times < -1,-1,0 > = < 1,-1,-2 >$ so plane is $(x-1) - (y-2) - 2z = 0$ or $x - y - 2z + 1 = 0$

Intersections

Is point $P(x_0, y_0, z_0)$ *on the line* $l(t) = < p_1 + tv_1, p_2 + tv_2, p_3 + t\,v_3 >$ Solve $x_0 = p_1 + tv_1$ for *t*, check that *t* against other components	*Is* $P(1,2,3)$ *on* $< 3-2t, 2t, 1-4t >$*?* $1 = 3 - 2t$ so that $t = 1$. Does not check out, though, so no.
Is point $P(x_0, y_0, z_0)$ *on the plane* $ax + by + cz + d = 0$ Plug $x = x_0, y = y_0, z = z_0$ into plane's equation	*Is* $P(1,2,3)$ *on* $2x - y + 3z = 5$*?* $2 \cdot 1 - 2 + 3 \cdot 3 = 9$ checks out so yes
Does line $l(t) = < p_1 + tv_1, p_2 + tv_2, p_3 + t\,v_3 >$ *intersect the plane* $ax + by + cz + d = 0$*?* They do unless they are parallel (check if $\vec{v} \cdot \vec{n} = 0$). To find intersection, plug $x = p_1 + tv_1, y = p_2 + tv_2, z = p_3 + tv_3$ into equation of plane and solve for *t*	*Does* $< 2t, -1+t, 3+4t >$ *intersect* $3x + 2y + z = 13$*?* Substituting line into plane: $3(2t) + 2(-1+t) + (3 + 4t) = 13$ simplifies to $t = 1$. So intersect at $l(1) = < 2,0,7 >$
Do two lines l_1 *and* l_2 *intersect?* Setup system of three equations $l_1(t) = l_2(s)$ with two unknowns. Solve first two for *s* and *t*, then check your answer against third equation	*Do* $l_1(t) = < 1-t, 2+2t, 3+t >$ *and* $l_2(t) = < t, 1-t, 3+2t >$ *meet* $l_1(t) = l_2(s)$ implies $1 - t = s, 2 + 2t = 1 - s, 3 + t = 3 + 2s$. Thus $2 + 2t = 1 - (1-t)$ or $t = -2$, so $s = -3$. Does NOT check 3rd eq $3 + t = 3 + 2s$ so lines do not intersect.
Do two planes intersect? If they are parallel (check if $n_1 \,\|\, n_2$, i.e. if $n_1 = c\,n_2$) they don't. Otherwise they intersect in a line $l(t) = P_0 + t\,\vec{v}$. The directional vector v is part of both planes, thus $v = n_1 \times n_2$. To find point P_0, assume $z = 0$ and solve remaining systems of two equations in two unknowns for x and y.	*Do* $x + y + z = 1$ *and* $2x - y + z = 0$ *intersect?* $\vec{n}_1 = < 1,1,1 >$ and $\vec{n}_2 = < 2,-1,1 >$. So $\vec{v} = \vec{n}_1 \times \vec{n}_2 = < 2,1,-3 >$ and to find $P(x, y, z = 0)$, solve $x + y = 1$ and $2x - y = 0$. Thus, $x = 1/3$ and $y = 2/3$. So $l(t) = < \frac{1}{3}, \frac{2}{3}, 0 > + < 2,-1,1 > t$