Math 2511: Calc III - Practice Exam 3

1. State the meaning or definitions of the following terms:
   a) vector field, conservative vector field, potential function of a vector field, volume, length of a curve, work, surface area, flux of a vector field, triple integral
   b) curl and divergence of a vector field F, gradient of a function

   look it up

   c) $\iint_R dA$ $= area(R)$

   $\iint_R f(x,y)dA$ $= \lim_{n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{n} f(x_i, y_j)\Delta x_i \Delta y_j = \int_a^b\int_c^d f(x,y)\,dy\,dx$

   $\iiint_Q f(x,y,z)dV$ $= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta x_i \Delta y_j \Delta z_k = \int_a^b\int_c^d\int_e^f f(x,y,z)\,dz\,dy\,dx$

   d) $\int_C ds$ ~~or~~ $= \int_a^b \sqrt{(x')^2+(y')^2}\,dt$

   $\int_C f(x,y)ds$ $= \int_a^b f(x(t),y(t))\sqrt{(x')^2+(y')^2}\,dt$

   $\int_C f(x,y)dx$ $= \int_a^b f(x(t),y(t))\,x'(t)\,dt$

   $\int_C f(x,y)dy$ $= \int_a^b f(x(t),y(t))\,y'(t)\,dt$

   e) $\int_C \vec{F}\cdot d\vec{r}$ $= \int_C M\,dx + N\,dy + P\,dz$

   $\int_C M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$

   f) $\iint_S g(x,y,z)\cdot dS$ $= \iint_R g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dA$, $S: z=f(x,y)$

   $\iint_S \vec{F}\cdot\vec{n}\,dS$ $= \iint_R -f_xM - f_yN + P\,dA$, with $z=f(x,y)$

   g) What does it mean when a "line integral is independent of the path"?

   If γ_1 and γ_2 are two paths, both starting at A and ending at B, then $\int_{\gamma_1} \vec{F}\,dr = \int_{\gamma_2} \vec{F}\,dr$

   h) State the Fundamental Theorem of Line Integrals. Make sure to know when it applies, and when it helps

   If C is a curve given by $\gamma: [a,b] \to \mathbb{R}^3$ and $\vec{F}$ a conservative vector field with potential function f, then
   $\int_C \vec{F}\,dr = f(\gamma(b)) - f(\gamma(a))$

   i) State Green's Theorem. Make sure to know when it applies, and in what situation it helps.

If C is a closed curve and $\vec{F} = \langle M, N \rangle$ a 2D vector field such that all partials are continuous, then

$$\oint_C \vec{F}\, d\vec{r} = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

j) State Stoke's Theorem. Make sure to know when it applies, and in what situation it helps.

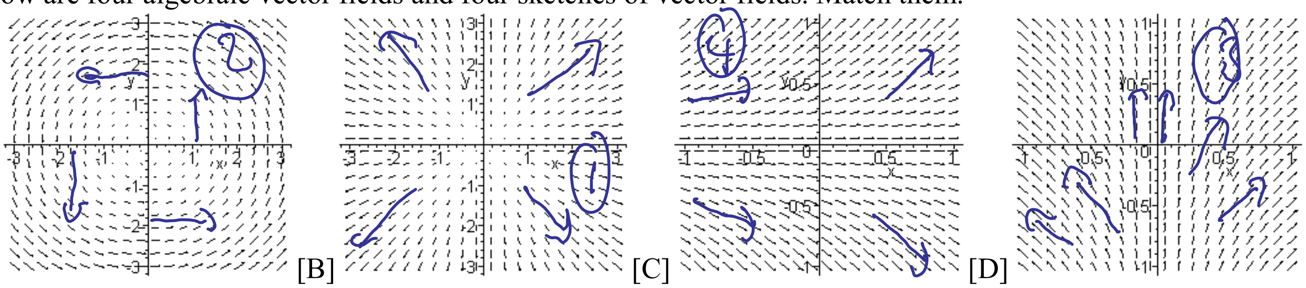
~~If S is a surface in $\mathbb{R}^3$ with a closed bounding curve C, all properly oriented, then $\oint_C \vec{F}\, d\vec{r} = \iint_S \mathrm{curl}(F)\cdot\vec{n}\, dS$~~

k) State Gauss' Theorem. Make sure to know when it applies, and in what situation it helps.

If S is a closed surface in $\mathbb{R}^3$ around Q then

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_Q \mathrm{div}(F)\, dV$$

2. Below are four algebraic vector fields and four sketches of vector fields. Match them.

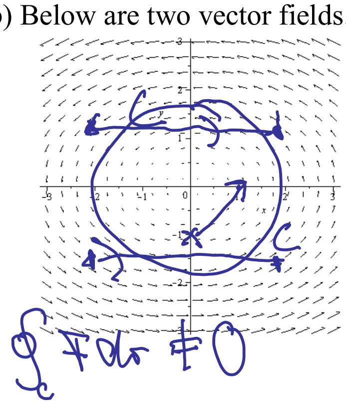


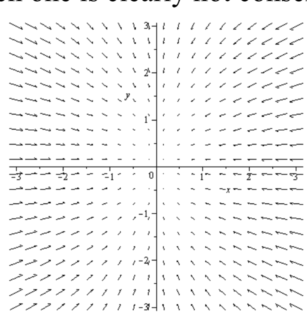
[A] [B] [C] [D]

(1) $F(x,y) = < x, y >$, (2) $F(x,y) = < -y, x >$, (3) $F(x,y) = < x, 1 >$, (4) $F(x,y) = < 1, y >$

B A D C

b) Below are two vector fields. Which one is clearly not conservative, and why?





So the 1st is not conservative

$\oint_C \vec{F}\, d\vec{r} \neq 0$

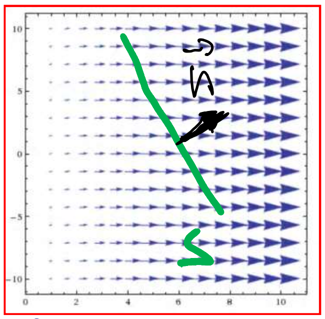
c) Say in the left vector field above you integrate over a straight line from (0,-1) to (1,0). Is the integral positive, negative, or zero? line and vectors are similar $\Rightarrow \int_\gamma \vec{F}\, d\vec{r} > 0$

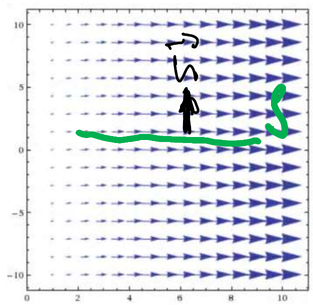
How about if you integrate from (-2,1) to (2,1)?

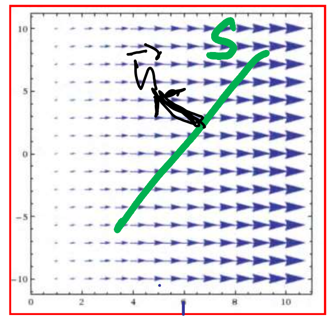
line moves against vectors $\Rightarrow \int_\gamma \vec{F}\, d\vec{r} < 0$

How about from (-2,-1) to (2,-1)? $\int_r \vec{F}\, d\vec{r} > 0$

d) Below are two slices of a 3D surface submerged in a 3D vector field. Imagine the picture as extending outwards of the paper. The normal vector shown is determined by the orientation of the surface. Is the flux $\iint_S \vec{F} \cdot \vec{n}\, dS$ positive, negative, or zero, for each surface?







$\vec{F} \cdot \vec{n} > 0$ ⇒ Flux positive

$\vec{F} \cdot \vec{n} = 0$ ⇒ Flux zero

$\vec{F} \cdot \vec{n} < 0$ ⇒ Flux negative

3. Are the following statements true or false:

a) If the divergence of a vector is zero, the vector field is conservative. F

b) If $F(x, y, z)$ is a conservative vector field then $curl(F) = 0$ T

c) If a line integral is independent of the path, then $\iint_S F \cdot n\, dS = 0$ for every path surface S F

d) If a vector field is conservative then $\int_C F \cdot dr = 0$ for every closed path C T

e) $\iint_R dA$ gives the surface area of the region R F

f) $\iint_R f(x, y)dA$ gives the volume of the region under the surface *f(x,y)* and over R, if *f* is positive. T

g) $\iiint_Q dV$ gives the volume of Q T

h) Can you apply the Fundamental Theorem of line integrals for the function $f(x, y, z) = xy\sin(z)\cos(x^2 + y^2)$?

No (need vector field)

i) Can you apply the Fundamental Theorem of line integrals for the vector field $F(x, y) =< 6xy^2 - 3x^2, 6x^2y + 3y^2 - 7 >$? $N_x = 12xy$, $M_y = 12xy$, so YES

j) Can you apply Green's theorem for a curve C, which is a straight line from (0,0,0) to (1,2,3)?

No (need closed curve)

k) Can you apply Green's theorem for a vector field $F =< xy, yz, zx >$ and a closed surface S?

No, but Gauss Thm would work

l) Can you apply Gauss' theorem for a vector field $F = < xy, yz, zx >$ and a surface S given by $z = f(x,y)$

No (need closed surface)

m) ~~Can you apply Stoke's theorem for a 3D vector field and a surface S given by z = f(x,y)~~

Maybe (depends on surface)

4. Suppose that $F(x,y,z) =< x^3y^2z, x^2z, x^2y >$ is some vector field. Find, if possible

a) *div(F),* $\text{div}\{x^3 y^2 z, x^2 z, x^2 y\} = 3x^2 y^2 z$

curl(F), $\text{curl}\{x^3 y^2 z, x^2 z, x^2 y\} = \{0, x\,y\,(x^2 y - 2), -2\,x\,z\,(x^2 y - 1)\}$

div(~~div~~(F)),

curl(curl(F)), $\text{curl curl}\{x^3 y^2 z, x^2 z, x^2 y\} = \{-2x^3 z, 2z(3x^2 y - 1), y(3x^2 y - 2)\}$

div(curl(F)), 0

cur~~l~~(div(F)),

grad(~~curl~~(F))

grad(div(F)) $\text{grad div}\{x^3 y^2 z, x^2 z, x^2 y\} = \{6x y^2 z, 6x^2 y z, 3x^2 y^2\}$

b) *grad., div.*, and *curl* of the vector field if appropriate for $< x^2, y^2, z^2 >$

grad d.n. apply

$\text{div}(F) = 2x + 2y + 2z$, $\text{curl}(F) = \langle 0, 0, 0 \rangle$

c) *grad., div.*, and *curl* of the vector field if appropriate for $< \cos(y) + y\cos(x), \sin(x) - x\sin(y), xyz >$

grad d.n. apply

$\text{div}(F) = -y\sin(x) - x\cos(y) + xy$, $\text{curl}(F) = \langle xz, -yz, 0 \rangle$

d) *grad., div.*, and *curl* of the vector field if appropriate for $f(x,y,z) = z\ln(x^2 + y^2)$

$\text{grad}(F) = \left\langle \frac{z\,2x}{x^2+y^2}, \frac{z \cdot 2y}{x^2+y^2}, \ln(x^2+y^2) \right\rangle$ curl and div d.n. apply

5. Decide which of the following vector fields are conservative. If a vector is conservative, find its potential function

a) $F(x,y) =< 2xy, x^2 >$

yes. $f = x^2 y + y x^2 + C$

b) $F(x,y) = < e^x \cos(y), e^x \sin(y) >$

No

c) $F(x,y,z) = < \sin(y), -x\cos y, 1 >$

No (close)

d) $F(x,y,z) = < 2xy, x^2 + z^2, 2zy >$

Yes, $f = x^2 y + y z^2 + C$

e) $F(x,y) = < 6xy^2 - 3x^2, 6x^2 y + 3y^2 - 7 >$

yes, $f(x,y) = 3x^2y^2 - x^3 + y^3 - 7y + C$

f) $F(x,y) = < -2y^3 \sin(2x), 3y^2(1+\cos(2x)) >$

yes $N_x = -6y^2 \sin(2x)$, $M_y = -6y^2 \sin(2x) \Rightarrow f = \cos(2x)\, y^3 + y^3 + C$

g) $F(x,y,z) = < 4xy + z, 2x^2 + 6y, 2z >$

No: $\mathrm{curl}(F) = <0,1,0>$

h) $F(x,y,z) = < 4xy + z^2, 2x^2 + 6yz, 2xz >$

No: $\mathrm{curl}\{4xy+z^2, 2x^2+6yz, 2xz\} = \{-6y, 0, 0\}$

6. Evaluate the following integrals:

a) $\iint_R \cos(x^2)\,dA$ where R is the triangular region bounded by y = 0, y = x, and x = 1

$$\int_0^1 \int_0^x \cos(x^2)\,dy\,dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(1)$$

b) $\int_0^1 \int_1^{2y} x^2 y^3\,dx\,dy$

$$\int_0^1 \int_1^{2y} x^2 y^3\,dx\,dy = \frac{25}{84}$$

c) $\int_C ds$, where C is the curve given by $r(t) = < t^2, 1+t >$, $0 \le t \le 2$ (you might want to use Maple at some point)

$$\int_0^2 \sqrt{(2t)^2 + 1}\,dt = \sqrt{17} + \frac{1}{4}\sinh^{-1}(4)$$

d) $\int_C x^2 y^3 dx$, where C is the curve given by $r(t) =< t^2, t^3 >$, $0 \le t \le 2$

$\int_0^2 (t^2)^2 (t^3)^3 \cdot 2t\,dt = 2\int_0^2 t^{14}\,dt = \frac{2}{15} 2^{15}$

e) $\int_C x^2 - y + 3ds$ where C is the circle $r(t) =< 2\cos(t), 2\sin(t) >$, $0 \le t \le \pi$

$$\int_0^{\pi} \left((2\cos(t))^2 - 2\sin(t) + 3\right) 2\,dt = 10\pi - 8$$

f) $\int_C x^2 - y + 3zds$ where C is a line segment given by $r(t) =< t, 2t, 3t >$, $0 \le t \le 1$

$$\int_0^1 \left(t^2 - 2t + 3\right)\sqrt{1 + 2^2 + 3^2}\,dt = \frac{7\sqrt{14}}{3}$$

g) $\int_C F \cdot dr$ where $F(x, y) =< y, x^2 >$ and C is the curve given by $r(t) =< 4 - t, 4t - t^2 >$, $0 \le t \le 3$

$$\int_0^3 \left(\left(4t - t^2\right)(-1) + (4 - t)^2 (4 - 2t)\right) dt = \frac{69}{2}$$

h) $\int_C F \cdot dr$ where $F(x, y) =< yz, x^2, zy >$ and C is the curve given by $r(t) =< 1 - t, 3t, 2 - t^2 >$, $1 \le t \le 3$

$$\int_1^3 \left((3t)\left(2 - t^2\right)(-1) + (1 - t)^2\, 3 + \left(2 - t^2\right)(3t)(-2t)\right) dt = \frac{1152}{5}$$

i) $\int_C ydx + x^2 dy$ where C is a parabolic arc given by $r(t) =< t, 1 - t^2 >$, $-1 \le t \le 1$

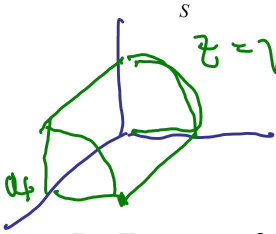
$$\int_{-1}^1 \left(\left(1 - t^2\right) 1 + t^2 (-2t)\right) dt = \frac{4}{3}$$

j) Find the surface integral $\iint_S x - 2y + zdS$, where S is the surface $z = 10 - 2x + 2y$ such that x is between 0 and 2 and y is between 0 and 4.

$$\int_0^4 \int_0^2 (x - 2y + (10 - 2x + 2y))\sqrt{1 + 4 + 4}\,dx\,dy = 216$$

k) $\iint_S (x+z)dS$ where S is the first-octant portion of the cylinder $y^2+z^2=9$ between x = 0 and x = 4

$z=\sqrt{9-y^2}$



$$\int_0^4\int_0^3\left(x+\sqrt{9-y^2}\right)\sqrt{1+0+\frac{y^2}{9-y^2}}\,dy\,dx = 12(3+\pi)$$

7. For some of the following line integrals there may be short-cut you can use to simplify your computations (but justify your shortcut by quoting the appropriate theorem)

a) $\int_C F\cdot dr$ where $F(x,y)=\langle e^x\cos(y), -e^x\sin(y)\rangle$ and C is the curve $r(t)=\langle 2\cos(t), 2\sin(t)\rangle$, $0\le t\le 2\pi$

C closed 2D ⇒ Green's: $\int F\,dr = \iint_D -e^x\sin(y)+e^x\sin(y)\,dA = 0$

(F is conservative)

b) $\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$ where C is some smooth curve from (0,0,0) to (1,4,3)

$\int F\,dr$, $F=\langle 2xyz, x^2z, x^2y\rangle$ F has potential $f = x^2yz$

$= x^2yz\Big|_{(0,0,0)}^{(1,4,3)} = 1\cdot4\cdot3 = 12$

c) $\int_C F\cdot dr$ where $F(x,y)=\langle y^3+1, 3xy^2+1\rangle$ and C is the upper half of the unit circle, from (1,0) to (-1,0)

$f = xy^3+x+y$ is potential

⇒ $\int F\,dr = xy^3+x+y\Big|_{(1,0)}^{(-1,0)} = -1-1 = -2$

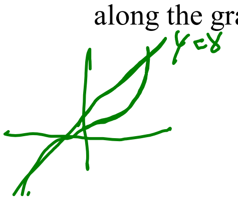
d) $\int_C F\cdot dr$ where $F(x,y)=\langle y^3x, 3xy^2\rangle$ and C is the line segment from (-1,0) to (2,3).

M N

$\frac{\partial N}{\partial x} = 3y^2$, $\frac{\partial M}{\partial y} = 3y^2x$ Not conservative $r(t) = \langle -1,0\rangle + t\langle 3,3\rangle = \langle -1+3t, 3t\rangle$

⇒ $\int F\,dr = \int_0^1 (3t)^3\cdot(3t-1)(3) + 3(3t-1)(3t)^2(3)\,dt = 45$

e) $\int_C y^3dx + (x^3+3xy^2)dy$ where C is the path from (0,0) to (1,1) along the graph of $y=x^3$ and from (1,1) to (0,0) along the graph of $y=x$.



closed curve ⇒ Green's $\oint_C F\,dr = \iint_R 3x^2+3y^2-3y^2\,dA = \iint_R 3x^2\,dy\,dx =$

$3\int_0^1\int_0 x^2\,dy\,dx = \frac{1}{4}$

8. Green's Theorem

a) Use Green's theorem to find $\int_C F \cdot dr$ where $F(x,y) = < y^3, x^3 + 3xy^2 >$ and C is the circle with radius 3, oriented counter-clockwise (You may need the double-angle formula for cos somewhere during your computations, or use Maple)

$$\int_C \vec{F}\, d\vec{r} = \iint 3x^2 + 3y^2 - 3y^2\, dA = \iint 3x^2\, dx\, dy =$$

circle

$$= \int_0^{2\pi} \int_0^{1} 3r^2 \cos^2\theta\, r\, dr\, d\theta = \frac{3\pi}{4}$$

b) Evaluate $\iint_R dA$ where R is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ by using a vector field $F(x,y) = < -\frac{y}{2}, \frac{x}{2} >$ and the boundary C of the ellipse R. Note that we did this in class, it is a very special application of Green's theorem.

c) Find the surface integral $\iint_S x - 2y + z dS$, where S is the surface $z = 10 - 2x + 2y$ such that x is between 0 and 2 and y is between 0 and 4.

$$\iint_R x - 2y + (10 - 2x + 2y)\sqrt{9}\, dA = \int_0^2 \int_0^4 10 - x\, dy\, dx = 72$$

d) Evaluate the flux integral $\iint_S \vec{F} \cdot \vec{n}\, dS$ where $F(x,y,z) = < x, y, z >$ and S is $x^2 + y^2 + z^2 = 4$

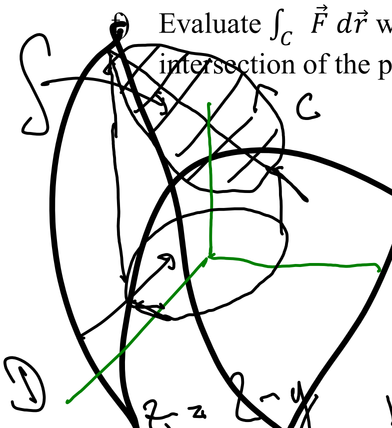
Gauss Thm Flux $= \iiint_{\text{sphere}} \operatorname{div}(\vec{F})\, dV = 3 \cdot \text{Volume (sphere)}$

$$= 3 \cdot \frac{4}{3}\pi 2^3 = 2^5 \pi$$

e) Evaluate $\int_C \vec{F}\, dr$ where $F(x,y,z) = < z^2, x^2, y^2 >$ and C is the boundary of the surface S given by $z = 4 - x^2 - y^2$ and $z \geq 0$, oriented counter-clockwise.

Stoke's $\oint \vec{F} ds = \iint \text{curl}(\vec{F})\, dA = 0$

f) Evaluate $\int_C \vec{F}\, d\vec{r}$ where $F(x,y,z) = < -y^2, x, z^2 >$ and C is the curve bounding the ellipse S consisting of the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$



$\int_C \vec{F}\, dr = \iint_S \text{curl}(\vec{F}) \cdot \vec{n}\, dS = \iint_D \langle 0, 0, 2y+1 \rangle \cdot \langle 0, 1, 1 \rangle\, dA$

$z = 2 - y$, $f_x = 0$, $f_y = -1$

$= \iint_{disk} 2y + 1\, dA = \int_0^{2\pi} \int_0^1 (2r\sin(t) + 1)\, r\, dr\, dt = \pi$

g) Evaluate $\iint_S curl(F)\, n\, dS$ where $F(x,y,z) = < xz, yz, xy >$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ above the xy-plane.

Use Stoke's backwards: $\iint_S \text{curl}(\vec{F}) \cdot \vec{n}\, dS = \int_C \vec{F}\, d\vec{r} =$

$C: r(t) = \langle \cos(t), \sin(t), k \rangle$

$= \int_C xz\, dx + yz\, dy + xy\, dz$

$= \int k\cos(t)(-\sin(t)) + k\sin(t)\cos(t) + 0\, dt = \underline{0}$

9. Evaluate the following integrals. You can use any theorem that's appropriate:

h) $\int_C 2xyz dx + x^2 z dy + x^2 y dz$ where C is a smooth curve from (0,0,0) to (1,4,3)

$f = x^2 y z$ is potential $\Rightarrow$

$\int \vec{F}\, dr = x^2 y z \Big|_{(0,0,0)}^{(1,4,3)} = \underline{12}$

i) $\int_C y dx + 2x dy$ where C is the boundary of the square with vertices (0,0), (0,2), (2,0), and (2,2)

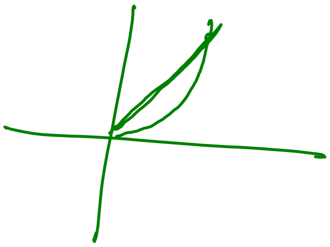
closed curve: Green: $\iint_{square} 2 - 1\, dA = \text{area(square)} = \underline{4}$

ellipse

j) $\int_C xy^2dx + x^2ydy$, where C is given by $r(t) =< 4\cos(t), 2\sin(t) >$, t between 0 and 2 Pi.

closed curve ⇒ Green: $\iint_D 2xy - 2yx \, dA = 0$

k) $\int_C xydx + x^2dy$ where C is the boundary of the region between the graphs of $y = x^2$ and $y = x$.



Green: $\iint_D 2x - x \, dA = \int_0^1 \int_{x^2}^{x} x \, dy \, dx = \frac{2}{15}$

10. Prove that if $F(x, y, z) =< M(x, y, z), N(x, y, z), P(x, y, z) >$ is any vector field where M, N, P are twice continuously differentiable then $div(curl(F)) = 0$

$$curl(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle P_y - N_z, -(P_x - M_z), N_x - M_y \rangle$$

$$div(curl(F)) = (P_y - N_z)_x - (P_x - M_z)_y + (N_x - M_y)_z =$$

$$= \cancel{P_{yx}} - \cancel{N_{zx}} - \cancel{P_{xy}} + \cancel{M_{zy}} + \cancel{N_{xz}} - \cancel{M_{yz}} = 0 \quad \#$$

Use Green's Theorem to prove that integrals of a conservative vector fields over closed curves are zero (assuming that the closed curve encloses a simply connected region and all conditions of Green's theorem are satisfied).

C closed ⇒ $\oint_C F \, dr = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dA$

F conservative ⇒ $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ ⇒ $\oint_C F \, dr = \iint_D 0 \, dA = 0$

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