Calc 2 Practice Exam Supplement

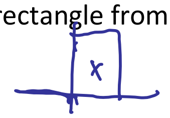
Here are a few supplementary questions involving mass, moments, and center of gravity. Any of these questions could appear on the exam 2 (or not).

1. Define the mass m, the moments M_x, M_y, and the center of gravity $(\bar{x}, \bar{y})$

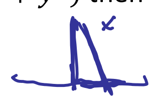
$$m = \iint_D \rho(x,y)\, dA \qquad (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

$$M_x = \iint_D y\, \rho(x,y)\, dA \qquad M_y = \iint_D x\, \rho(x,y)\, dA$$

2. True/False
   a. If D is a lamina of uniform density in form of a rectangle from $(0,0)$ to $(2,4)$, then the center of gravity $(\bar{x}, \bar{y}) = (1,2)$ True

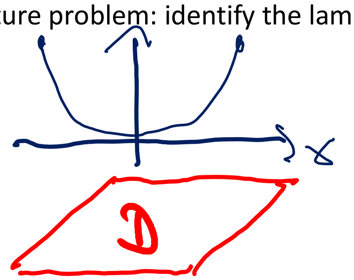


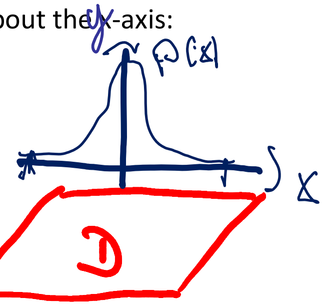
   b. If D is a lamina bounded by the x-axis, the y-axis, and the line from (0, 2) to (1, 0) with density function $\rho(x,y) = x^2 \sin(y) * \cos(x^2 + y^2)$ then the center of gravity is (1, 2)



False (can't be outside)

3. Picture problem: identify the lamina with the larger moment about the y-axis:





X more mass far away from y-axis

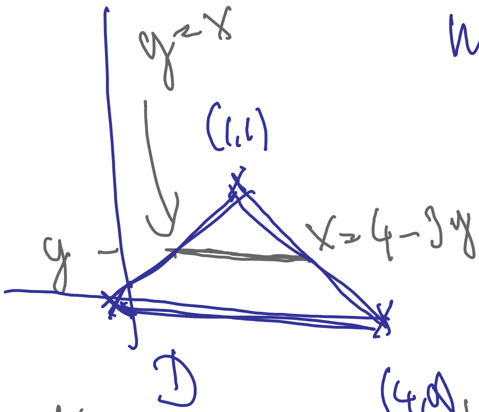
4. Find the mass and center of gravity of the lamina that occupies the region D and has the given density function:
   a. $D = \{(x,y) : 0 \le x \le 2, -1 \le y \le 1\}$ and $\rho(x,y) = xy^2$

$$m = \int_0^2 \int_{-1}^1 x y^2\, dy\, dx = 4/3$$

$$M_x = \int_0^2 \int_{-1}^1 x y^3\, dy\, dx = 0 \qquad (\bar{x}, \bar{y}) = \left(\frac{4}{3}, 0\right)$$

$$M_y = \int_0^2 \int_{-1}^1 x^2 y^2\, dy\, dx = 16/9$$

b. D is the triangular region with vertices (0,0), (1,1), (4,0) and $\rho(x,y) = x$

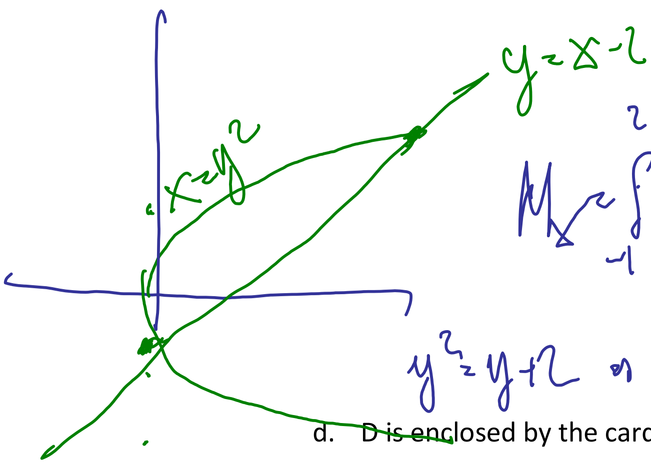


$$m = \iint_D x\,dA = \int_0^1 \int_y^{4-3y} x \,dx\,dy = \frac{10}{3}$$

$$M_x = \int_0^1 \int_y^{4-3y} y\,x\,dx\,dy = 1, \quad M_y = \int_0^1 \int_y^{4-3y} x^2\,dx\,dy = 7$$

$$m = -1/3, \quad y - 0 = -\frac{1}{3}(x-4), \quad y = -\frac{1}{3}x + \frac{4}{3} \quad \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{4}{10}, \frac{3}{10}\right)$$

c. D is bounded by the parabola $x = y^2$ and the line $y = x - 2$, and $\rho(x,y) = 3$

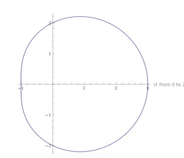


$$m = \int_{-1}^{2} \int_{y^2}^{y+2} 3\,dx\,dy$$

$$M_x = \int_{-1}^{2} \int_{y^2}^{y+2} 3y\,dx\,dy = \quad , \quad M_y = \int_{-1}^{2} \int_{y^2}^{y+2} 3x\,dx\,\cancel{dy} =$$

$$y^2 = y + 2 \Rightarrow y = 2, -1$$

d. D is enclosed by the cardioid $r = 1 + \cos(\theta)$ and $\rho(x,y) = \sqrt{x^2 + y^2}$



$$m = \int_0^{2\pi} \int_0^{1+\cos\theta} \sqrt{r^2}\, r\,dr\,d\theta = \frac{5\pi}{3}$$

$$M_x = \int_0^{2\pi} \int_0^{1+\cos\theta} r\sin\theta \sqrt{r^2}\, r\,dr\,d\theta = 0$$

$$M_y = \int_0^{2\pi} \int_0^{1+\cos\theta} r\cos\theta \sqrt{r^2}\, r\,dr\,d\theta = \frac{7\pi}{4}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\frac{7\pi}{4}}{\frac{5\pi}{3}}, 0\right) = \left(\frac{7\pi}{4} \cdot \frac{3}{5\pi}, 0\right) = \left(\frac{21}{20}, 0\right)$$

e) D bounded by $y=\sqrt{2-x^2}$, $y=0$ and $\rho(x,y)=1/(x^2+y^2+1)$

$$m=\iint_D \rho(x,y)\,dA = \int_0^{\pi}\int_0^{\sqrt{2}} \frac{1}{r^2+1}\, r\,dr\,d\theta =$$

$$= \int_0^{\pi}\int_0^{\sqrt{2}} \frac{r}{1+r^2}\,dr\,d\theta \underbrace{=}_{\text{Mathematica}} \frac{1}{2}\pi\ln(3)$$

$$M_x=\iint_D y\,\frac{1}{x^2+y^2+1}\,dA = \int_0^{\pi}\int_0^{\sqrt{2}} \frac{r^2\sin(\theta)}{r^2+1}\,dr\,d\theta \underbrace{=}_{\text{Mathematica}} 2\left(\sqrt{2}-\tan^{-1}(\sqrt{2})\right)$$

$$M_y=\iint_D x\,\frac{1}{x^2+y^2+1}\,dA = \int_0^{\pi}\int_0^{\sqrt{2}} \frac{r^2\cos(\theta)}{r^2+1}\,dr\,d\theta = 0$$

$$(\bar{x},\bar{y})=\left(\frac{M_x}{m},\frac{M_y}{m}\right)=\left(\frac{2(\sqrt{2}-\tan^{-1}(\sqrt{2}))}{\frac{1}{2}\pi\ln(3)},\,0\right)=\underline{\underline{(0.53184,0)}}$$