Math 2411 – Calc III Practice Exam 2

This is a practice exam. The actual exam consists of questions of the type found in this practice exam, but will be shorter. If you have questions do not hesitate to send me email. Answers will be posted if possible – no guarantee.

1. **Definitions**: Please state in your own words the following definitions:
   a) Limit of a function $z = f(x, y)$
   b) Continuity of a function $z = f(x, y)$
   c) partial derivative of a function f(x,y)
   d) gradient and its properties
   e) directional derivative of a function f(x, y) in the direction of a unit vector u
   f) The (definition and geometric meaning of) the double integral of *f* over the region *R* $\iint_R f(x, y)dA$
   [redacted]

2. **Theorems:** Describe, in your own words, the following:
   [redacted]
   b) the procedure to find relative extrema of a function f(x, y)
   c) the procedure to find absolute extrema of a function f(x, y)
   d) how to switch a double integral to polar coordinates
   e) a theorem that allows you to evaluate a double integral easily

3. **True/False** questions:
   a) If $\lim_{(x,y)\to(0,0)} f(x, y) = 0$ then $\lim_{x\to 0} f(x, 0) = 0$ True. If general limit exists, the more specific one does, too.

   b) If $\lim_{y\to 0} f(0, y) = 0$ then $\lim_{(x,y)\to(0,0)} f(x, y) = 0$ False. If $\lim_{y\to 0} f(0,y)=0$ anything is possible for general limit

   c) $\lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h} = \frac{\partial}{\partial x} f(x, y)$ False
   Σ definition of $D_{\hat{u}}(P)$

   d) If f is continuous at (0,0), and f(0,0) = 10, then $\lim_{(x,y)\to(0,0)} f(x, y) = 10$ True by the very definition of continuity

   [redacted]

   g) If f(x, y) is a function such that all second order partials exist and are continuous then $f_{xx} = f_{yy}$ False $f_{xy} = f_{yx}$ but not f_{xx} and f_{yy}

   h) The volume under f(x,y), where $a \le x \le b$ and $g(x) \le y \le h(x)$ is $\int_a^b \int_{g(x)}^{g(y)} f(x, y)dxdy$ False

   i) If *f(x,y)* is continuous then $\int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$ True - Fubini's thm

j) If $f(x,y)$ is continuous then $\int_a^b\int_c^d f(x)g(y)dydx = \left(\int_a^b f(x)dx\right)\cdot\left(\int_c^d g(y)dy\right)$ True. If f only depends on x g only on y it works

k) If f is continuous over a region D then $\iint_D f(x,y)\,dxdy = \iint_D f(r,\theta)\theta\, d\theta dr$ False $r\,dr\,d\theta$

4. **Surfaces**: Find the domain for the following functions

a) $f(x,y) = \dfrac{1}{xy}$ $xy \neq 0 \Rightarrow$ domain : $\mathbb{R}^2 - \{x=0 \text{ or } y=0\}$

b) $f(x,y) = \dfrac{1}{x^2+y^2}$ $x^2+y^2=0 \Rightarrow x=y=0 \Rightarrow$ domain : $\mathbb{R}^2 - \{(0,0)\}$

c) $f(x,y) = \dfrac{1}{x^2-y^2}$ $x^2-y^2=0 \Rightarrow x=\pm y \Rightarrow$ domain : $\mathbb{R}^2 - \{x=\pm y\}$

5. **Limits and Continuity**: Determine the following limits as $(x,y) \to (0,0)$, if they exist.

a) $\lim_{(x,y)\to(0,0)} \dfrac{xy+1}{x^2+y^2+1} = 1/1 = 1$

b) $\lim_{(x,y)\to(0,0)} \dfrac{xy+1}{x^2+y^2} = 1/0$ undefined

c) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$: $x=0$: $\lim_{y\to0} \dfrac{0}{y^2} = 0$ } different so dne

d.n.e.

$x=y$: $\lim_{x\to0} \dfrac{x^2}{2x^2} = \frac{1}{2}$

d) $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$

Note: $\dfrac{x^2}{x^2+y^2} < 1 \Rightarrow \left|\dfrac{x^2y}{x^2+y^2}\right| < |y| < \sqrt{x^2+y^2}$

Thus: Given $\varepsilon > 0$ pick $\delta = \varepsilon$. Then if

$\|(x,y)\| < \delta \Rightarrow \sqrt{x^2+y^2} < \delta = \varepsilon$

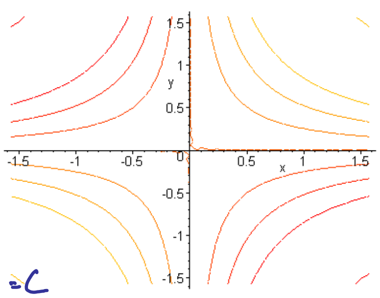
$\Rightarrow \left|\dfrac{x^2y}{x^2+y^2}\right| < \sqrt{x^2+y^2} < \varepsilon$

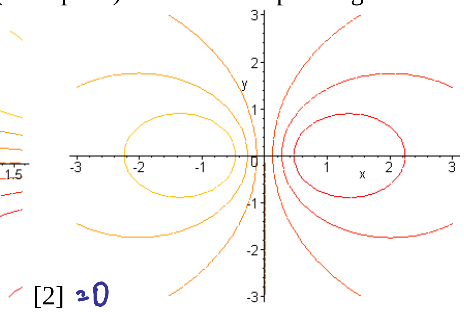
$\Rightarrow |f(x,y)| < \varepsilon \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$

e) $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ $x=0$: $\lim_{y\to 0} \frac{-y^2}{y^2} = -1$
$y=0$: $\lim_{x\to 0} \frac{x^2}{x^2} = 1$
} different so d.n.e.

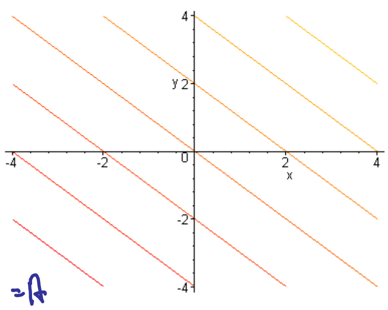
d.n.e.

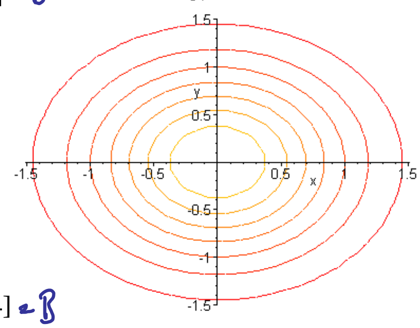
6. **Picture**: Match the following contour plots (level plots) to their corresponding surfaces.



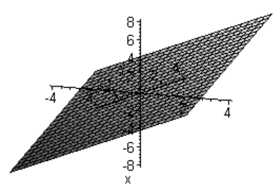


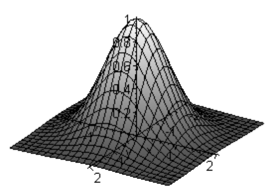
d) [1] = C [2] = D



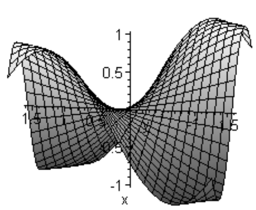


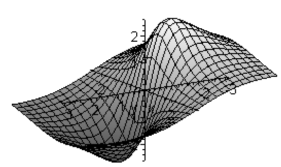
e) [3] = A [4] = B





f) [A] = ③ [B] = ④





g) [C] = ① [D] = ②

Other picture problems:

- Given a contour plot, draw the gradient vector at specific points
- classify some regions as type-1, type-2, or neither.

7. **Differentiation**: Find the indicated derivatives for the given function:

a) Find

b) Suppose $f(x,y) = 2x^3y^2 + 2y + 4x$, find

f_x $= 6x^2y^2 + 4$

f_y $= 4x^3y + 2$

f_{xx} $= 12xy^2$

f_{xy} $= 12x^2y$

f_{yy}, $= 4x^3$

match

f_{yx} $= 12x^2y$

[redacted]

d) Let $f(x,y) = y^2e^x + y$. Find

$f_x = y^2e^x$, $f_{xx} = y^2e^x$, $f_y = 2ye^x + 1$, $f_{yy} = 2e^x$, $f_{xy} = f_{yx} = 2ye^x$

f_{xyy} $= (f_{xy})_y = 2e^x$

$\leftarrow$ because $(f_{xy}) = (f_{yx}) \Rightarrow (f_{xy})_y = (f_{yx})_y$!

f_{yxy} $= (f_{yx})_y = 2e^x$

f_{yyx} $= 2e^x$

8. **Directional Derivatives**:

a) Find the directional derivative of f(x, y) = xy e^{xy} at (-2, 0) in the direction of a vector u, where u makes an angle of Pi/4 with the x-axis.

$u = \langle \cos(\pi/4), \sin(\pi/4) \rangle = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

$f_x = y e^{xy} + xy\, y e^{xy} = y e^{xy} + xy^2 e^{xy} \Rightarrow$ at $(-2,0)$: $f_x = 0$

$f_y = x e^{xy} + xy\, x e^{xy} = x e^{xy} + x^2 y e^{xy} \Rightarrow$ at $(-2,0)$: $f_y = -2$

$\Rightarrow D_f(u) = \langle 0, -2 \rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle = \underline{\underline{-\sqrt{2}}}$

b) Find $D_u(f)$ where $f(x, y) = \frac{x}{y} - \frac{y}{x}$ and $\vec{u} = < -\frac{4}{5}, \frac{3}{5} >$

$f_x = \frac{1}{y} + \frac{y}{x^2}$

$f_y = -\frac{x}{y^2} - \frac{1}{x}$

$\Rightarrow D_{\vec{u}}(f) = \langle \frac{1}{y} + \frac{y}{x^2}, -\frac{x}{y^2} - \frac{1}{x} \rangle \cdot \langle -\frac{4}{5}, \frac{3}{5} \rangle = \underline{-\frac{4}{5}\left(\frac{1}{y} + \frac{y}{x^2}\right) - \frac{3}{5}\left(\frac{x}{y^2} + \frac{1}{x}\right)}$

c) Suppose $f(x, y) = x^2 e^y$. Find the maximum value of the directional derivative at (-2, 0) and compute a unit vector in that direction.

max dir. deriv. is $\|\nabla f\|$

$f_x = 2x e^y$

$f_y = x^2 e^y$

at $(-2,0)$: $f_x = -4$, $f_y = 4$

$\Rightarrow \|\nabla f\| = \|\sqrt{(-4)^2 + (4)^2}\| = \underline{\sqrt{32}}$

$\Rightarrow$ unit vector in that dir. $\underline{\frac{1}{\sqrt{32}} \langle -4, 4 \rangle}$

10. **Max/Min Problems:** Compute the extrema as indicated

a) $f(x, y) = 3x^2 - 2xy + y^2 - 8y$. Find relative extreme and saddle point(s), if any.

$f_x = 6x - 2y = 0$

$f_y = -2x + 2y - 8 = 0$

$4x - 8 = 0$

$\Rightarrow x = 2, y = 6$ is critical point.

$H = \begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}$, $D = 12 - 4 = 8$

$D > 0$, $f_{xx} > 0 \Rightarrow (2,6)$ is minimum

b) $f(x, y) = 4xy - x^4 - y^4$. Find relative extrema and saddle point(s), if any

$f_x = 4y - 4x^3 = 0 \Rightarrow y = x^3$

$f_y = 4x - 4y^3 = 0 \Rightarrow x - x^9 = 0$

$x(1 - x^8) = 0 \Rightarrow x = 0, 1, -1$

$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

$x = -1 \Rightarrow y = -1$

3 critical points

$(0,0)$, $(1,1)$, $(-1,-1)$

$H = \begin{pmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{pmatrix}$; $D = 144x^2y^2 - 16$

at $(0,0)$: $D < 0 \Rightarrow$ saddle point at $(0,0)$

at $(1,1)$: $D > 0$, $f_{xx} < 0 \Rightarrow$ max at $(1,1)$

at $(-1,-1)$: $D > 0$, $f_{xx} < 0 \Rightarrow$ max at $(-1,-1)$

c) Let $f(x,y)=3xy-6x-3y+7$. Find absolute maximum and minimum inside the triangular region spanned by the points (0,0), (3, 0), and (0, 5). $f_x = 3y-6=0$, $f_y = 3x-3=0 \Rightarrow (1,2)$ critical

$x=0$: $f(x,y) = -3y+7$ no crit.

$y=0$: $f(x,y) = -6x+7$ no crit.

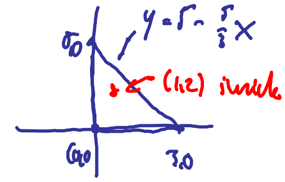
$y = 5-\frac{5}{3}x$: $f(x) = -5x^2+14x-8$

$f'(x) = -10x+14 = 0$

$x = 7/5,\ y = 8/3$

the 3rd leg of the triangle

(x,y)	$f(x,y)$	
$(1,2)$	1	
$(\frac{7}{5}, \frac{8}{3})$	$9/5$	
$(0,0)$	7	← abs. max
$(3,0)$	-11	← abs. min
$(0,5)$	-8	



~~b)~~ d) Let $f(x,y)=3x^2-2xy+y^2-8y$. Find the absolute extrema over [0, 1] x [0, 2]

critical is $(\cancel{2,6})$ – not in $[0,1] \times [0,2]$

$x=0$: $f(y) = y^2-8y \to f' = 2y-8 \to \cancel{y=4}$

$x=1$: $3-2y+y^2-8y = y^2-10y+3 \to \cancel{y=5}$

$y=0$: $3x^2 \to x=0$

$y=2$: $3x^2-4x+4-16 \Rightarrow f' = 6x-4,\ x = \frac{2}{3}$ ✓

$(0,0)$	0	
$(1,0)$	3	← abs. max
$(0,2)$	-12	
$(1,2)$	-13	
$(\frac{2}{3}, 2)$	-13.33	← abs. min

11. Evaluate the following integrals:

a) $\int_0^1\int_0^2 xy^2\,dx\,dy = \int_0^1 \frac{1}{2}x^2y^2\Big|_{x=0}^{2} dy = \int_0^1 2y^2\,dy = \frac{2}{3}y^3\Big|_0^1 = \frac{2}{3}$

b) $\int_0^{\pi}\int_0^{\pi/2} \sin(x)\cos(y)\,dy\,dx = \int_0^{\pi} \sin(y)\sin(x)\Big|_{y=0}^{y=\pi/2} dx = \int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi} = -\cos(\pi)+\cos(0) = 2$

c) $\int_0^2\int_{x^2}^{x} (x^2+2y)\,dy\,dx = \int_0^2 yx^2+y^2\Big|_{y=x^2}^{y=x} dx = \int_0^2 (x-x^2)x^2 + x^2 - x^4\,dx =$

$= \int_0^2 x^3-x^4+x^2-x^4\,dx = \frac{1}{4}(2)^4 - \frac{2}{5}(2)^5 + \frac{1}{3}(2^3) =$

$= 4 - \frac{64}{5} + \frac{8}{3} = -\frac{92}{15}$

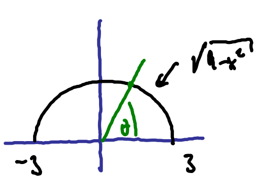
d) $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^3 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\; r\,dr\,d\theta =$

$x = r\cos\theta$

$y = r\sin\theta$

$dy\,dx = r\,dr\,d\theta$

$\int_0^{\pi}\int_0^3 r^2\,dr\,d\theta = \frac{1}{3}r^3\Big|_0^3 \cdot \pi = 9\pi$



e) $\int_0^a\int_0^b\int_0^c \cancel{x^2+y^2+z^2\,dxdydz} = \int_0^1\int_0^1 x\sin(xy)\,dxdy$ or $\int_0^1\int_0^1 x\sin(xy)\,dydx$. Let's try either one.

$$\int_0^1\int_0^1 x\sin(xy)\,dxdy = \int_0^1 -\frac{1}{y}\cos(xy)\Big|_{x=0}^{x=1} dy = \int_0^1 -\frac{1}{y}\cos(y)+\frac{1}{y}\,dy = \int_0^1 \frac{1}{y}-\frac{\cos(y)}{y}\,dy$$

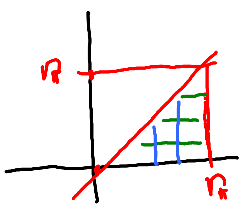
tricky to integrate!

error!!

$$\int_0^1\int_0^1 x\sin(xy)\,dydx = \int_0^1 \cancel{x}\cdot\left(-\frac{1}{\cancel{x}}\cos(xy)\right)\Big|_{y=0}^{y=1} dx = \int_0^1 -\cos(x)+1\,dx = -\sin(x)+x\Big|_0^1 = \underline{-\sin(1)+1}$$

changed

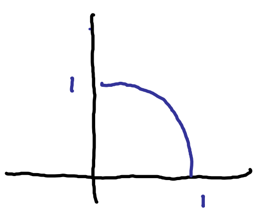
f) $\int_0^{\sqrt{\pi}}\int_y^{\cancel{x}\ \sqrt{\pi}} \cos(x^2)\,dxdy = \int_0^{\sqrt{\pi}}\int_0^{x} \cos(x^2)\,dydx = \int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big|_0^{\sqrt{\pi}} = \frac{1}{2}(\sin(\pi)-\sin(0)) = \underline{0}$



originally: for a y between 0 and $\sqrt{\pi}$, x goes horizontally from $x=y$ to $x=\sqrt{\pi}$

changed dx, dy: x is fixed between 0 and $\sqrt{\pi}$, while y goes vertically from $y=0$ to $y=x$

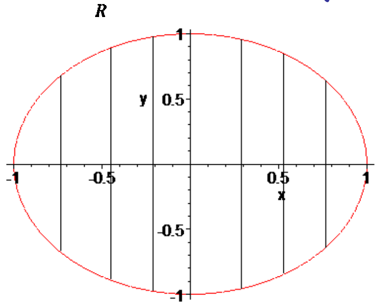
g) $\iint_R \sqrt{x^2+y^2}\,dA$, where R is the part of the unit circle in the 1[st] quadrant



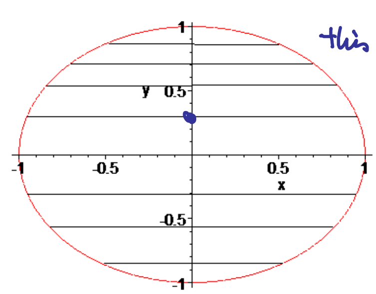
$$\iint_R \sqrt{x^2+y^2}\,dA = \int_0^{\pi/2}\int_0^1 r\cdot r\,dr\,d\theta \quad \text{in polar coords.}$$

$$= \frac{1}{3}\cdot\frac{\pi}{2} = \underline{\frac{\pi}{6}}$$

12. The pictures below show to different ways that a region R in the plane can be covered. Which picture corresponds to the integral $\iint_R f(x,y)\,dxdy$ fix y then x



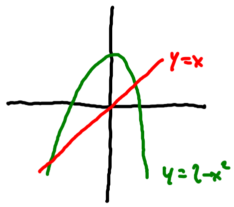
this is $\iint f\,dy\,dx$



this is right

changed this

13. Suppose you want to evaluate $\iint_R f(x,y)dA$ where R is the region in the xy plane bounded by ~~$y \geq 0$~~, $y = 2 - x^2$, and $y = x$. According to Fubini's theorem you could use either the iterated integral $\iint f(x,y)dxdy$ or $\iint f(x,y)dydx$ to evaluate the double integral. Which version do you prefer? Explain. You do not need to actually work out the integrals.

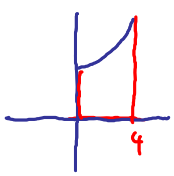


$\iint f\, dx\, dy$ would result in 2 integrals

$\underline{\iint f\, dy\, dx}$ is one integral therefore $\underline{\text{simpler}}$

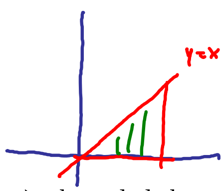
14. Use a multiple integral and a convenient coordinate system to find the volume of the solid:

a) bounded by $z = x^2 - y + 4$, $z = 0$, $y = 0, x = 0$, and $x = 4$



$y = x^2+4$ $\quad V = \int_0^4 \int_0^{x^2+4} x^2 - y + 4 \, dy\, dx = \underline{3296/15}$ Maple

b) bounded by $z = e^{-x^2}$ and the planes $y = 0$, $y = x$, and $x = 1$



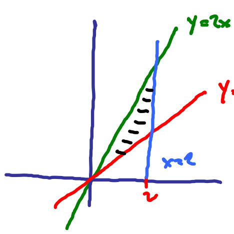
$V = \int_0^1 \int_0^x e^{x^2}\, dy\, dx = \underline{\tfrac{1}{2}(e-1)}$ Maple

c) bounded above by $z = \sqrt{16 - x^2 - y^2}$ and bounded below by the circle $x^2 + y^2 \leq 4$

$$\iint_R \sqrt{16-x^2-y^2}\, dA = \int_0^{2\pi}\int_0^2 \underbrace{\sqrt{16-r^2}\, r}_{(16-r^2)^{1/2}\cdot r}\, dr\, d\theta = \int_0^{2\pi} -\frac{2}{3}\frac{1}{2}(16-r^2)^{3/2}\Big|_{r=0}^{2} \cdot d\theta = \underline{2\pi\left(\frac{64}{3} - 8\sqrt{3}\right)}$$

Maple

d) evaluate $\iint_R \frac{y}{x^2+y^2}$ where R is a triangle bounded by $y = x$, $y = 2x$, $x = 2$



$$\int_0^1\int_x^{2x} \frac{y}{x^2+y^2}\, dy\, dx = \int_0^1 \frac{1}{2}\ln(x^2+y^2)\Big|_{y=x}^{y=2x} dx = \int_0^1 \frac{1}{2}\ln(5x^2) - \frac{1}{2}\ln(2x^2)\, dx$$

$$= \frac{1}{2}\int_0^1 \ln(5) + 2\ln(x) - \ln(2) - 2\ln(x)\, dx = \frac{1}{2}\int_0^1 \ln(5) - \ln(2)\, dx =$$

$$= \underline{\frac{1}{2}(\ln(5) - \ln(2))}$$

e) bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy plane

15. Find the following surface areas:

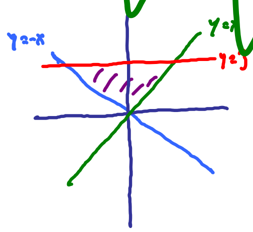
a) of the plane $z = 2 - x - y$ above the rectangle $0 \le x \le 2$ and $0 \le y \le 3$

$f_x = -1,\ f_y = -1 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{3}$

$\Rightarrow$ area $= \int_0^2 \int_0^3 \sqrt{3}\, dy\, dx = \underline{\underline{6\sqrt{3}}}$

b) of the cylinder $z = 9 - y^2$ above the triangle bounded by $y = x$, $y = -x$, and $y = 3$

$f_x = 0,\ f_y = -2y \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4y^2 + 1}$



$\int_0^3 \int_{-y}^{y} \sqrt{4y^2+1}\, dx\, dy = \int_0^3 x\sqrt{4y^2+1}\Big|_{x=-y}^{x=y} dy = 2\int_0^3 y\sqrt{4y^2+1}\, dy =$

$= 2 \cdot \frac{2}{3} \cdot \frac{1}{8} (4y^2+1)^{3/2} \Big|_0^3 = \frac{37}{6}\sqrt{37} - \frac{1}{6}$

c) of the surface $z = 16 - x^2 - y^2$ above the circle $x^2 + y^2 \le 9$ disk radius 3

$f_x = -2x,\ f_y = -2y \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$

Polar coordinates: $A = \int_0^{2\pi} \int_0^3 \sqrt{4r^2+1}\, r\, dr\, d\theta \overset{\text{simple}}{=} \left(\frac{37}{6}\sqrt{37} - \frac{1}{6}\right)\pi$

16. **Prove** the following facts:

a) Use the definition to find f_x for $f(x, y) = xy$ of course I know $f_x = y$. Need to prove it:

$f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = \lim_{h \to 0} \frac{(x+h)y - xy}{h} = \lim_{h \to 0} y\frac{(x+h-x)}{h} = \lim_{h \to 0} y\frac{h}{h} = \lim_{h \to 0} y = \underline{\underline{y}}$

b) Use the definition to find f_x for $f(x, y) = xy$

same problem

c) A function f is said to satisfy the Laplace equation if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Show that the function $f(x, y) = \ln(x^2 + y^2)$ satisfies the Laplace equation.

$f_x = \frac{2x}{x^2+y^2},\ f_{xx} = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}$

$f_{yy} = \frac{2x^2 - 2y^2}{(x^2+y^2)^2} \Rightarrow f_{xx} + f_{yy} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2} + \frac{2x^2 - 2y^2}{(x^2+y^2)^2} = 0$ so $\underline{\text{true}}$

d) Two function u(x, y) and v(x, y) are said to satisfy the Cauchy-Riemann equations if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Show that the functions $u(x,y) = e^x \cos(y)$ and $v(x,y) = e^x \sin(y)$ satisfy the Cauchy-Riemann equations.

$u_x = e^x \cos(y)$ $\quad u_y = -e^x \sin(y)$ $\quad \Rightarrow u_x = v_y$ and

$v_x = e^x \sin(y)$ $\quad v_y = e^x \cos(y)$ $\quad u_y = -v_x$ $\quad$ so true

e) Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$ Then show that f has partial derivatives at (0, 0) but f is not differentiable at (0, 0) – hard!

Not applicable

f) Prove that the volume of a sphere with radius R is 4/3 * Pi * ~~r~~ R^3 Sphere: $x^2+y^2+z^2=R^2 \Rightarrow z = \pm\sqrt{R^2-x^2-y^2}$

$$V = 2\cdot\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\,dy\,dx = 2\cdot\int_0^{2\pi}\int_0^R \sqrt{R^2-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi} -\frac{2}{3}\cdot\frac{1}{2}\left(R^2-r^2\right)^{3/2}\Big|_{r=0}^{r=R}\,d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}\left(0-(R^2)^{3/2}\right)\right]d\theta = \int_0^{2\pi}\frac{2}{3}R^3\,d\theta = \frac{4}{3}\pi R^3$$

g) Prove that the surface area of a sphere with radius R is 4 * Pi * ~~r~~ R^2

$z = \sqrt{R^2-x^2-y^2} \Rightarrow f_x = \frac{-x}{\sqrt{R^2-x^2-y^2}}$, $f_y = \frac{-y}{\sqrt{R^2-x^2-y^2}}$

$\Rightarrow f_x^2 + f_y^2 + 1 = \frac{x^2}{R^2-x^2-y^2} + \frac{y^2}{R^2-x^2-y^2} + 1 = \frac{x^2+y^2+R^2-x^2-y^2}{R^2-x^2-y^2} = \frac{R^2}{R^2-x^2-y^2}$

$\Rightarrow$ area $= 2\iint_{Disk}\sqrt{\frac{R^2}{R^2-x^2-y^2}}\,dA = 2\int_0^{2\pi}\int_0^R \frac{R}{\sqrt{R^2-r^2}}\,r\,dr\,d\theta$

$= 2R\int_0^{2\pi}\int_0^R \frac{r}{\sqrt{R^2-r^2}}\,dr\,d\theta = 2R\int_0^{2\pi} -\left(R^2-r^2\right)^{1/2}\Big|_0^R\,d\theta =$

$= -2R\int_0^{2\pi} -\left(R^2\right)^{1/2}\,d\theta = -2R\cdot R\cdot 2\pi = 4\pi R^2$