Panel 1

Last time

Contour integrals of $f(x,y)$, $f(x,y,z)$ — $\int_\gamma f(x,y)\,ds$, $\int_\gamma f(x,y)\,dx$, dy, dz

Work (Contour integrals of vector fields)

$$\int_\gamma \vec{F}\,d\vec{r} = \int_\gamma M\,dx + N\,dy + P\,dz$$

$F = \langle M, N, P \rangle$

Panel 2

Fundamental Thm of Contour Integration:

If $\vec{F}$ is a conservative vector field with potential function $f(x,y,z)$ and $\gamma : [a,b] \to \mathbb{R}^3$ a smooth curve then:

$$\int_\gamma \vec{F}\,d\vec{r} = f(\gamma(b)) - f(\gamma(a))$$

Consequences: F is conservative. Then

(1) $\int_{\gamma_1} \vec{F}\,dr = \int_{\gamma_2} \vec{F}\,dr$, γ_1, γ_2 have same start/end

(2) $\oint_\gamma \vec{F}\,dr = 0$ $\quad \gamma$ any closed curve!

Panel 3

Ex: Let $F(x,y) = \left\langle \frac{y^2}{1+x^2},\ 2y\arctan(x) \right\rangle$ and

$r(t) = \langle t^2, 2t \rangle$, $t \in [0,1]$. Find $\int_\gamma \vec{F}\, d\vec{r}$

Old way: $\int_\gamma \vec{F}\, d\vec{r} = \int_\gamma \frac{y^2}{1+x^2}\, dx + 2y\, \text{atan}(x)\, dy =$

$= \int_0^1 \frac{4t^2}{1+t^4}\, 2t\, dt + 2\cdot 2t \arctan(t^2)\, 2dt$

New way: $\vec{F}$ conservative? $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ✓

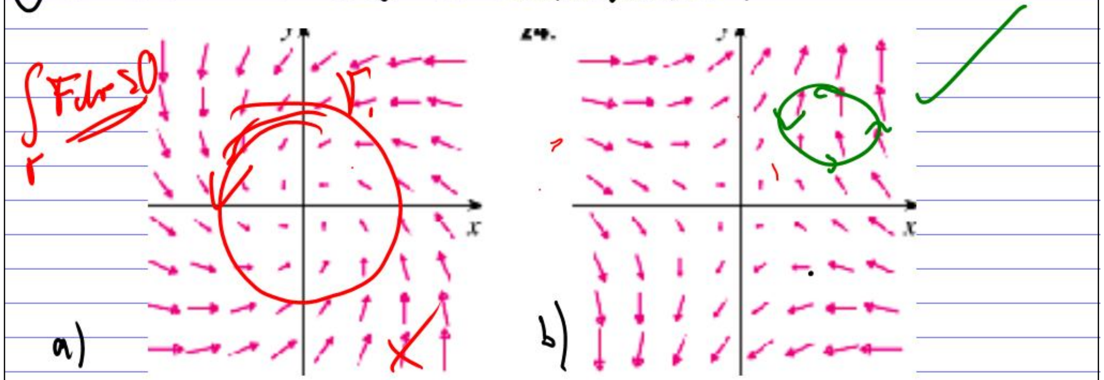
Potential $f_x = \frac{y^2}{1+x^2} \Rightarrow f(x,y) = y^2 \arctan(x)$, $f_y = N$ ✓

$\Rightarrow \int_\gamma \vec{F}\, d\vec{r} = y^2\, \text{atan}(x) \Big|_{(0,0)}^{(1,2)} = \underline{4\, \text{atan}(1)}$

Panel 4

Quiz #10 (Preview) Name: ______

(1) Which vector field is conservative?



(2) Is $\int_C y^2 - 1\, dx + 2xy\, dy$ independent of the path C from A to B? YES NO

$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$

Panel 5

(3) Find the work done moving a particle along a straight line from (0,1) to (3,2) through the vector field $\vec{F}(x,y) = \langle 3x^2+y^2, 2xy \rangle$

F conservative with potential $f(x,y) = x^3 + xy^2$

$$\int_\gamma \vec{F}\,d\vec{r} = x^3 + xy^2 \Big|_{(0,1)}^{(3,2)}$$

(4) Find $\oint_C \tan(y)\,dx + x\sec^2(y)\,dy$ where $\gamma(t) = \langle \cos(t), \sin(t) \rangle,\ t \in [0, 2\pi]$

$\vec{F}$ is conservative, $\oint_\gamma \vec{F}\,d\vec{r} = 0$

Panel 6

Summary

$\int_\gamma \vec{F}\,d\vec{r}$ is important

$\int_\gamma \vec{F}\,d\vec{r}$
- long way: $r(t) = \langle x(t), y(t) \rangle$ ...
- if F conservative, shortcut

$\oint_\gamma \vec{F}\,d\vec{r}$
- long way
- zero (if conservative)
- Green's Theorem

Panel 7

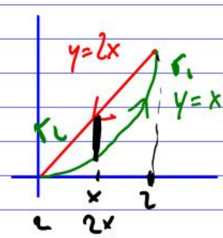
Green's Theorem: R a region in xy-plane with boundary curve C. C is piecewise smooth, non-intersecting, closed, and positively oriented. $\vec{F} = (M, N)$ is a smooth 2D vector field. Then:

$$\oint_C \vec{F}\, d\vec{r} = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

positive oriented: inside is to your left!

Panel 8

Ex: Evaluate $\oint_C 5xy\,dx + x^3\,dy$, where C is as shown:



Method A: Green's theorem applies

$$\oint 5xy\,dx + x^3\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

$$= \int_0^2 \int_{x^2}^{2x} 3x^2 - 5x\, dy\, dx = -\frac{28}{15}$$

Method B:

$$\int_0^2 5t\,t^2\,dt + t^3\,2t\,dt + \int_2^0 10t^2\,dt + t^3\,2\,dt = -\frac{28}{15}$$

$r_1(t) = \langle t, t^2 \rangle$, $t \in [0, 2]$

$r_2(t) = \langle t, 2t \rangle$, $t \in [2, 0]$

Panel 9

Ex: Evaluate $\oint_C \underbrace{2xy}_{M}\,dx + \underbrace{(x^2+y^2)}_{N}\,dy$, C is $4x^2+9y^2=36$

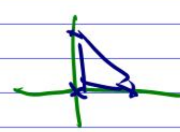
By Green:
$$= \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA =$$
$$= \iint_R 2x-2x\,dA = 0$$

Panel 10

Ex: Find $\oint_\gamma \overbrace{(x\sin(y^2)-y)}^{M}\,dx + \overbrace{(x^2y\cos(y^2)+3x)}^{N}$

where γ is the triangle $(0,0)$, $(1,0)$, $(0,1)$.

Green: $\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA =$



$$= \iint (\cancel{2xy\cos(y^2)}+3) - (\cancel{2xy\cos(y^2)}-1)\,dA =$$

$$= \iint_R 4\,dA = 4\iint_R dA = 4\cdot\text{area}(R) = 4\cdot\tfrac{1}{2} = 2$$

Panel 11

Evaluate $\oint_C \underbrace{(3y - e^{\sin(x)})}\,dx + \underbrace{(7x + \sqrt{y^4+1})}\,dy$

where C is the circle $x^2+y^2=9$

$$\iint 7 - 3 \, dA = \underline{\underline{4 \cdot \pi 9 =}}$$

Panel 12

Theorem: If D is a region enclosed by a closed curve C, pos. or

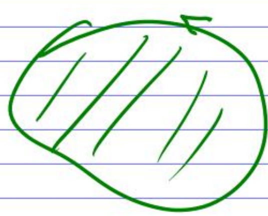
then $\text{area}(D) = \frac{1}{2}\oint_C x\,dy - y\,dx$

Odd way: Green's thm reversed

$$\oint \vec{F} \cdot d\vec{r} = \iint ? \, dA$$

$$1-(-1)$$

Proof $\oint x\,dy - y\,dx = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA =$



D

$M = -y$

$N = x$

$= 2\,\text{area}(D)$

Panel 13

Ex: Find area enclosed by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

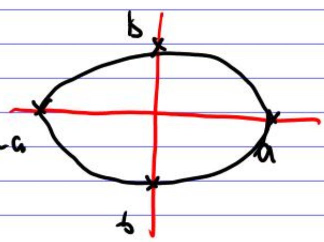
Already did: area of circle is πr^2 ✓

length of circle is $2\pi r$ ✓

volume of sphere is $\frac{4}{3}\pi r^3$ ✓

surface area of sphere $4\pi r^2$ X still

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$r(t) = \langle a\cos(t), b\sin(t)\rangle$

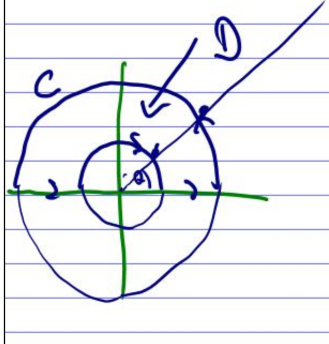
$= \frac{1}{2}\int_0^{2\pi} ab\, dt = \underline{\underline{\pi ab}}$

$\text{area}(D) = \frac{1}{2}\oint x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} ab\cos^2(t) + ab\sin^2(t)\,dt$

Panel 14

Ex: Evaluate $\oint_C y^2\,dx + 3xy\,dy$ where C is the boundary of the region between $x^2+y^2=1$ and $x^2+y^2=4$.

$\oint_C \overset{M}{y^2}\,dx + \overset{N}{3xy}\,dy = \iint_D 3y - 2y\,dA = \iint_D y\,dA =$



$= \int_0^{2\pi}\int_1^2 r\sin(\theta)\, r\,dr\,d\theta$

Panel 15

Alternate version of Green's Theorem:

Green's theorem only works in 2D

$F = \langle M, N \rangle$ ~ F in 3D is $F = \langle M, N, 0 \rangle$

Then $\text{curl}(F) = \left\langle 0, 0, \frac{\partial N}{\partial y} - \frac{\partial M}{\partial x} \right\rangle$

$$\int \vec{F}\, d\vec{r} = \iint_D \text{curl}(F) \cdot \vec{k}\, dA$$ — 2D thm with div(F)

3D Green's Thm use curl(F)

Panel 16

15. Evaluate $\int_C 2(x+y)dx + 2(x+y)dy$, C curve from (-2, 2) to (4, 3)

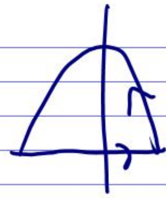
Fund. Thm

16. Find the work done by the force field $F = \langle 9x^2y^2, 6x^3y - 1 \rangle$ from P(0,0) to Q(5,9)

Fund Thm.

Panel 17

18. Evaluate $\oint_C 2xy\,dx + (x+y)\,dy$ where C bounds the region between $y=0$ and $y=4-x^2$.



Green ✓

21. Evaluate $\oint_C x\sin(y^2) - y^2)dx + (x^2\cos(y^2) + 3x)dy$ where C is the boundary of the trapezoid with vertices (0, -2), (1, -1), (1, 1), and (0, 2).

Green ✓