Panel 1

Review: Contour Integrals

$\int_\gamma f\,ds$, $\int_\gamma f\,dx$, $\int_\gamma f\,dy$, $\int_\gamma f\,dx + g\,dy$

Step 1: $r(t) = \langle \quad , \quad \rangle$ is γ

$\int \vec{F}\cdot d\vec{r}$, $\vec{F} = \langle f, g \rangle$

Work

Ex: $\int_\gamma y^3\,ds = \int_0^2 t\sqrt{9t^4+1}\,dt$ | γ: $x = t^3$, $y = t$, $t \in [0,2]$

$x = y^3$

$r(t) = \langle t^3, t \rangle$

Panel 2

$ds = \sqrt{(x')^2 + (y')^2}\,dt$ $\quad \langle t^3, t \rangle$

$dx = x'\,dt$, $\quad dy = y'\,dt$ $\quad x' = 3t^2$, $y' = 1$

$\int_\gamma x e^y\,dx$, γ: $x = e^y$, $(1,0)$ to $(e,1)$ ✓

$r(t) = \langle e^t, t \rangle$, $t \in [0,1]$

$\int_0^1 e^t \cdot e^t \cdot e^t\,dt = \int_0^1 e^{3t}\,dt = \frac{1}{3}e^{3t}\Big|_0^1 = \frac{1}{3}(e^3 - 1)$

Panel 3

$\int xy\,dx + (y-x)\,dy$, line from $(2,0)$ to $(3,2)$

$r(t) = \langle 2,0\rangle + t(\langle 3,2\rangle - \langle 2,0\rangle)$

$= \langle 2,0\rangle + t\langle 1,2\rangle \quad t\in[0,1]$

$= \langle 2+t,\ 2t\rangle$

$\quad x \quad\quad y$

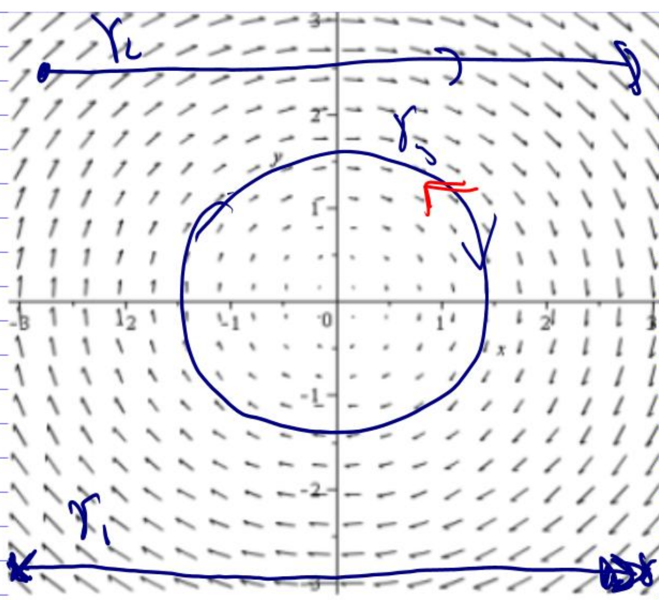
$\int_0^1 (2+t)(2t)\,dt + (2t-(2+t))\,2\,dt =$

$= \#$

$\int \vec F\, d\vec r$

$F = \langle xy,\ y-x\rangle$. Find work done by F along path γ

Panel 4



$\int_{\gamma_1} \vec F\, d\vec r < 0$

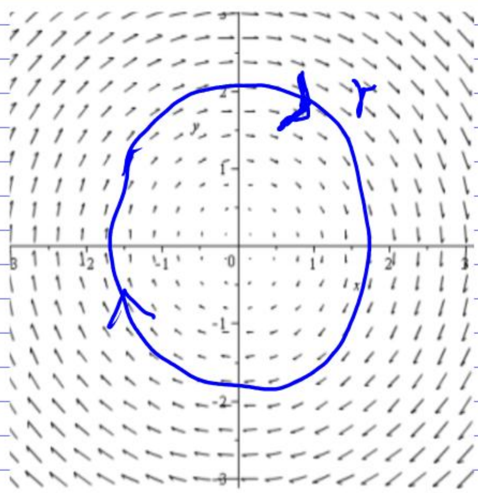
$\int_{\gamma_2} \vec F\, d\vec r > 0$

$\int_{\gamma_3} \vec F\, d\vec r > 0$ < 0

Panel 5

Name: __________

Quiz #9



(1) Is $\int_\gamma \vec{F} \, d\vec{r}$

a) positive

b) negative

c) zero

Panel 6

(2) Suppose γ is the line from $(-1,2)$ to $(2,3)$

a) $\int_\gamma x \, ds$

b) $\int_\gamma y \, dx$

c) $\int_\gamma y \, dx + x \, dy$

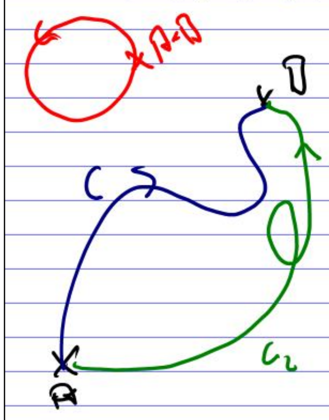
Panel 7

Fundamental Theorem of Line Integration

Suppose $\vec{F}$ is a conservative vector field. Then

$$\int_C \vec{F}\, d\vec{r} = f(B) - f(A),$$

where f is the potential function, C curve from A to B



Consequences (1) If $\vec{F}$ conservative, then $\int_C \vec{F}\, d\vec{r}$ is independent of the path C

(2) If C is closed, $\vec{F}$ conservative,

$$\Rightarrow \oint_C \vec{F}\, d\vec{r} = 0 \quad (= f(B) - f(B))$$

Panel 8

Ex: Find work done by gravitational field $\vec{F}(\vec{r}) = -\frac{mMG}{\|r\|^3}\vec{r}$ moving particle from $(3,4,12)$ to $(2,2,0)$.

Old way: Find parametrization of curve $\Rightarrow$ No curve given. Thus, work must only depend on start/end!

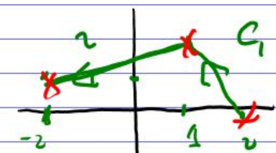
$\Rightarrow F(x,y,z) = \frac{-mMG}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$ needs to be conservative

Potential Function: $m \cdot M \cdot G\,(x^2+y^2+z^2)^{-1/2} = f(x,y,z)$

$\Rightarrow$ Work $f(2,2,0) - f(3,4,12) = mMG\left(8^{-1/2} - 169^{-1/2}\right)$

Panel 9

Evaluate $\int_C y^2\,dx + 2xy\,dy$.



① $C_1: r(t) = \langle 2,0\rangle + (\langle 1,2\rangle - \langle 2,0\rangle)t =$

$= \langle 2,0\rangle + t\langle -1,2\rangle = \langle 2-t, 2t\rangle, \quad t \in (0,1)$

$\int_0^1 -(2t)^2\,dt + 2(2-t)(2t)\cdot 2\,dt = \underline{4}$

$= \langle 1-3t, 2-t\rangle$

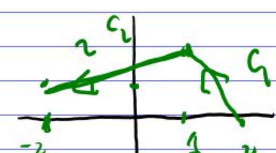
② $r(t) = \langle 1,2\rangle + (\langle -2,1\rangle - \langle 1,2\rangle)t = \langle 1,2\rangle + t\langle -3,-1\rangle$

$\int_0^1 (2-t)^2(-3)\,dt + 2(1-3t)(2-t)(-1)\,dt = \underline{-6}$

Answer: -2

Panel 10

Evaluate $\int_C y^2\,dx + 2xy\,dy$.



$= \int \vec{F}\,d\vec{r}, \quad \vec{F} = \langle y^2, 2xy\rangle \quad \Rightarrow \quad f(x,y) = xy^2$ ✓

$C_1: \int_{C_1} F\,dr = xy^2\Big|_{(2,0)}^{(1,2)} = \underline{\underline{4}}$

$C_2: \int_{C_2} F\,dr = xy^2\Big|_{(1,2)}^{(-2,1)} = -2-4 = -6$

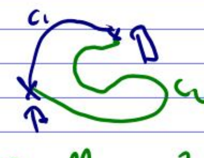
$\int_C F\,dr = xy^2\Big|_{(2,0)}^{(-2,1)} = \underline{-2}$ because of path independence.

Panel 11

<u>Thm:</u> If $\vec{F} = \nabla f$, i.e. $\vec{F}$ is conservative and C is a smooth curve from A to B then

$$\int_C \vec{F}\, d\vec{r} = f(B) - f(A)$$

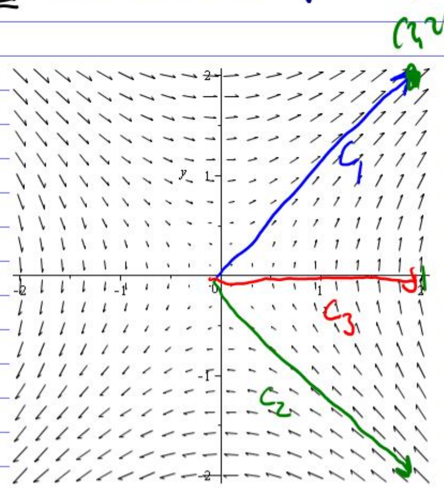
<u>Corollary 1:</u> $\int_{C_1} F\,dr = \int_{C_2} F\,dr$ if C_1, C_2 start and end at the same point, resp. and F conservative



<u>Corollary 2:</u> $\oint_C F\,dr = 0$ C closed, F conservative

Panel 12

<u>Ex:</u> Are the following integrals positive or negative?



$\int_{C_1} \vec{F}\,dr > 0$ (4)

$\int_{C_2} F\,dr = 0$ (0)

$\int_{C_3} F\,dr < 0$ (-4)

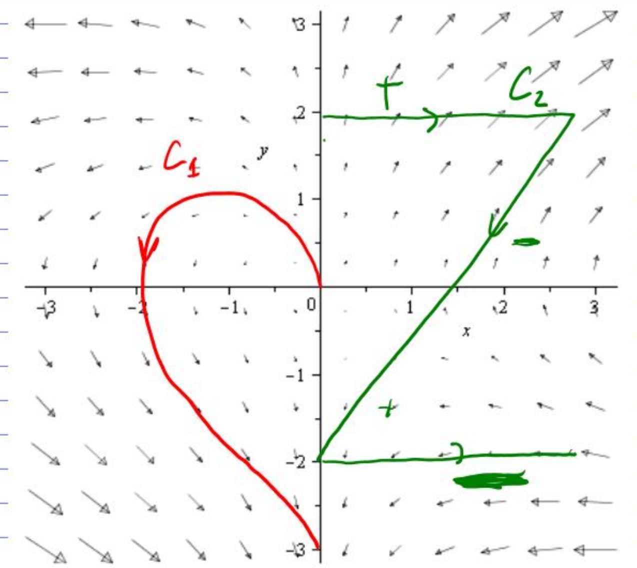
$\vec{F} = \langle y, x\rangle$

HW Verify

$C_1: r(t) = \langle 2,2\rangle t$

Panel 13

Ex: Are the following integrals positive or negative?



$\int_{C_1} \vec{F} \cdot d\vec{r}$

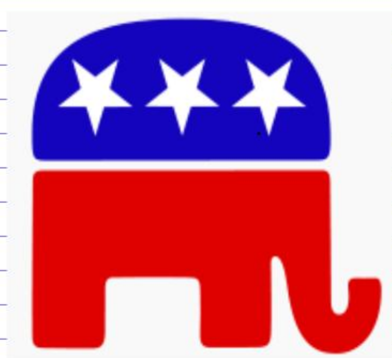
pos or neg

$\int_{C_2} \vec{F} \cdot d\vec{r}$

pos or neg

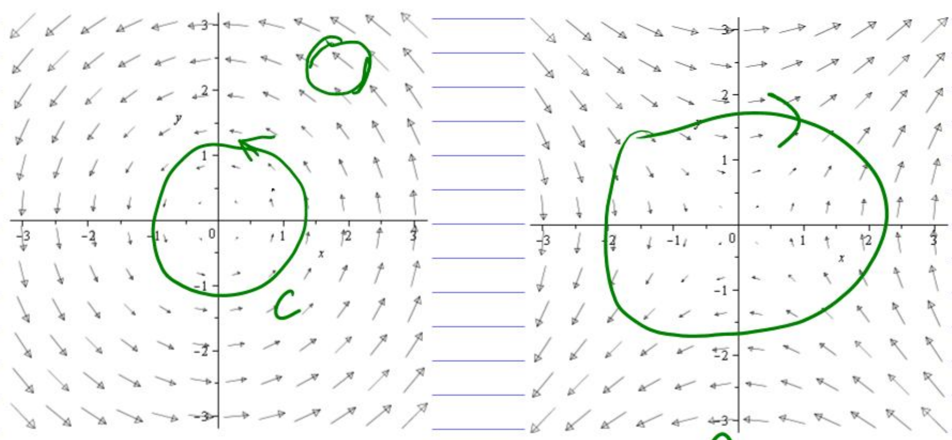
Panel 14

Which of the following vector fields look conservative?



Panel 15

Which of the following vector fields looks conservative?



$\oint_C F\,dr \neq 0$ is cons.

Not conservative

Panel 16

No class 4/18

Find a conservative vector field that has the given potential:

$f(z,y,z) = \sin(x^2 + y^2 + z^2)$

Find $(\nabla \cdot F)$ and $curl(F) = \nabla \times F$

$F(x,y,z) = < x^2 z, y^2 x, y + 2z >$

Evaluate $\int_C \underbrace{(x-y)}_{M}dx + \underbrace{x}_{N}dy$ if C is the graph of $y^2 = x$ from (4,-2) to (4,2)

Find the work done by F(x,y,z) along the curve $< t, t^2, t^3 >$ from (0, 0, 0) to (2, 4, 8), where $F(x,y,z) = < y, z, x >$

Check which of the following vector fields is not conservative.

$F(x,y) = < 3x^2 y + 2, x^3 + 4y^3 >$

$F(x,y) = < e^x, 3 - e^x \sin(y) >$

$F(x,y,z) = < 8xz, 1 - 6yz^2, 4x^2 - 9y^2 z^z >$

Show that the line integrals are independent of the path, and find their value:

$\int_{(-1,2)}^{(3,11)} (y^2 + 2xy)dx + (x^2 + 2xy)dx$

$\int_{(1,0,2)}^{(-2,1,3)} (6xy^3 + 2z^2)dx + (9x^2 y^2)dy + (4xz + 1)dz$

HW