Panel 1

Last Time:

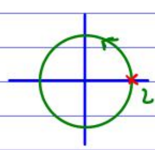
- Finding potential functions for conservative vector fields $\vec{F} = \langle M, N \rangle$ or $\vec{F} = \langle M, N, P \rangle$
- Line Integral of a function with respect to s along a curve γ $\int_\gamma f(x,y)\,ds$

(1) Find parametrization of γ. $r(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$

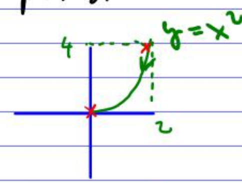
(2) $\int_\gamma f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

Panel 2

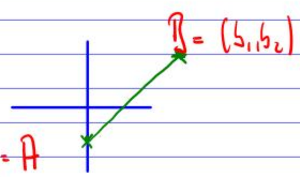
Before we continue, need to find paths. Find $r(t) = \langle x(t), y(t) \rangle$ expressions for these paths:



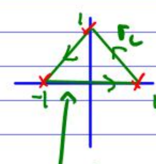
$r(t) = \langle 2\cos(t), 2\sin(t) \rangle$
$t \in [0, 2\pi)$



$r(t) = \langle t, t^2 \rangle$, $t \in [2, 0]$



$r(t) = A + (B - A)t$, $t \in [0,1]$



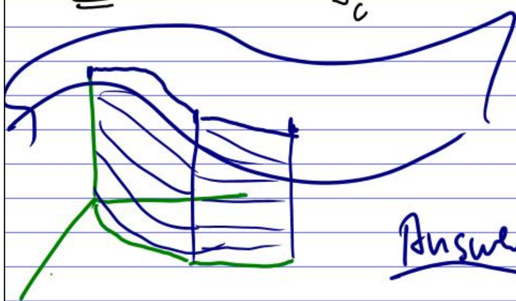
$r_1(t) = \langle -1, 0 \rangle + t\langle 2, 0 \rangle$

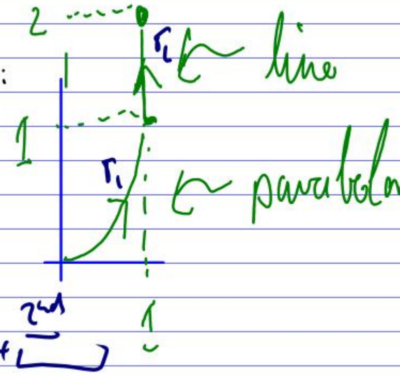
$r_2(t) = \langle 1, 0 \rangle + t\langle -1, 1 \rangle$

$r_3(t) =$ ____

Panel 3

Ex: Evaluate $\int_C 2x\,ds$ where C:





Answer $\underbrace{\quad}_{1^{st}} + \underbrace{\quad}_{2^{nd}}$

$r_1(t) = \langle \overset{x}{t}, \overset{y}{t^2} \rangle$, $t \in [0,1]$: $\int_{C_1} 2x\,ds = \int_0^1 2\cdot t\sqrt{1+(2t)^2}\,dt = \_\_$

$r_2(t) = \langle 1,1 \rangle + t\langle 1,0 \rangle$: $\int_{C_2} 2x\,ds = \int_0^1 2(1+t)\sqrt{1^2+0^2}\,dt$

$= \langle \underset{x}{1+t}, \underset{y}{1+0} \rangle$

$= \underline{\qquad}$

Panel 4

Note: If $f(x,y)=1$ then $\int_\gamma 1\,ds = \int_a^b \sqrt{(x')^2+(y')^2}\,dt$ gives length of curve γ

Now let's define 3 variations of line integral:

$\mathbb{R}^3$: $\int_C f(x,y,z)\,ds = \int f(x(t),y(t),z(t))\sqrt{x'^2+y'^2+z'^2}\,dt$ $r(t) = \langle x(t), y(t), z(t) \rangle$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Panel 5

Ex: Find $\int_C xy^2\,dx$ and $\int_C xy^2\,dy$ where C is parabola from $(0,0)$ to $(2,4)$. Also find $\int_C xy^2\,ds$

$$\int_0^1 xy^2\,dx = \frac{1}{2}x^2y^2\Big|_{x=0}^{1} = \frac{1}{2}y^2$$ Not our problem!

$C: r(t) = \langle t, t^2\rangle$, $t \in [0,2]$

$$\int_C xy^2\,dx = \int_0^2 t\cdot(t^2)^2\cdot 1\,dt = \int_0^2 t^5\,dt = \frac{1}{6}2^6$$

$$\int_C xy^2\,dy = \int_0^2 t(t^2)^2\,2t\,dt = \int_0^2 2t^6\,dt = \frac{2}{7}2^7$$

Panel 6

Ex: Evaluate $\int_{C_i} xy\,dx + x^2\,dy$ if

$C_1: \langle 3t-1,\ 3t^2-2t\rangle,\ 1 \le t \le 5/3$

C_2: line segment from $(2,1)$ to $(4,5)$

C_1: $$\int_1^{5/3} (3t-1)\cdot(3t^2-2t)\,3\,dt + (3t-1)^2(6t-2)\,dt = \_\_$$

C_2: $$\int_0^1 (2+2t)(1+4t)\,2\,dt + (2+2t)^2\,4\,dt = \_\_$$

$C_2: r(t) = \langle 2,1\rangle + t\langle 2,4\rangle,\ t \in [0,1]$

$= \langle 2+2t,\ 1+4t\rangle$

Panel 7

__Def:__ C a curve $r(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$

$$\Rightarrow \int_C f(x,y)\, \boxed{ds} = \int f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt$$

(line integral of f along curve C)

$$\Rightarrow \int_C f(x,y)\, dx = \int f(x(t), y(t))\, x'(t)\, dt$$

(line integral of f along curve C with respect to x)

$$\Rightarrow \int_C f(x,y)\, dy = \int f(x(t), y(t))\, y'(t)\, dt$$

(line integral of f along curve C with respect to y)

$$\Rightarrow \int_C f(x,y)\, dx + g(x,y)\, dy = \int_C f\, dx + \int_C g\, dy$$

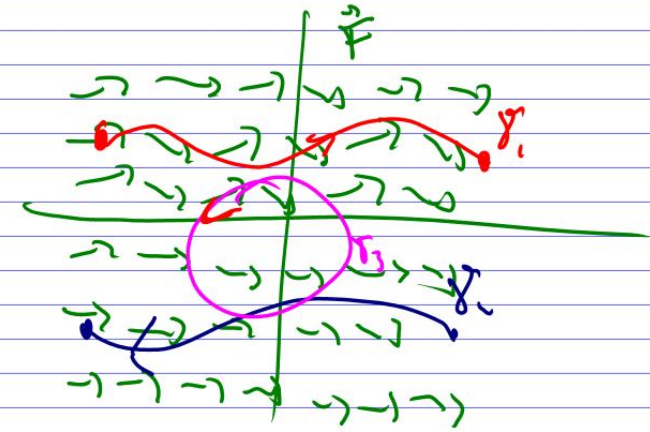
Panel 8

__Note:__ If $F(x,y) = \langle M(x,y), N(x,y) \rangle$ is a vector field and γ a curve parametrized by $r(t) = \langle x(t), y(t) \rangle$, then

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_\gamma \langle M, N \rangle \cdot \langle dx, dy \rangle = \int_\gamma M\, dx + N\, dy$$

is the __Work__ done along curve γ through vector field $\vec{F}$

Panel 9



$$\int_{\gamma_1} \vec{F}\, d\vec{r} > 0$$

$$\int_{\gamma_3} \vec{F}\, d\vec{r} \approx 0$$

$$\int_{\gamma_2} \vec{F}\, d\vec{r} < 0$$

Panel 10

Ex: Find $\int_C y^2\,dx + x\,dy$ where (a) C_1 is line from $(-5,-3)$ to $(0,2)$ and (b) C_2: $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$

$\vec{F}(x,y) = \langle y^2, x\rangle$ Then $\int_C \vec{F}\, d\vec{r} =$

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 (-3+5t)^2\, 5 + (-5+5t)\, 5\, dt$$

C_1: $r(t) = \langle -5,-3\rangle + t\langle 5,5\rangle$
$= \langle -5+5t, -3+5t\rangle$

$$= \int_0^1 \left((9 - 30t + 25t^2)5 + (-25+25t)\right) dt =$$

$= \#$

Panel 11

Integral Soup

$\int_a^b f(x)\,dx$ calc 1
(net) area under curve

$\iint_R f(x,y)\,dA$
(net) volume under curve

$\iiint_Q f(x,y,z)\,dV$???

$\int_C ds$ length of C

$\iint_R dS$ surface area - later

$\int_C f(x,y)\,ds$
"curtain"

$\int_C f(x,y)\,dx$ no meaning

$\int_C f(x,y)\,dy$ no meaning

$\int_C \vec{F}\,d\vec{r}$ Work

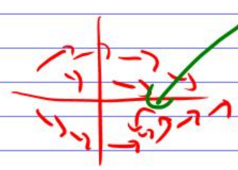
Panel 12

Let $f(x,y) = x^2 - xy + y^2$, $F(x,y) = < 2x - y, 2y - x >$, $\boxed{D = \{(x,y) : x^2 + y^2 \le 1\}}$, $C = \{(x,y) : x^2 + y^2 = 1, y \ge 0\}$, $\gamma_1(t) = < t, 0 >$, $t \in [-1,1]$, and $\gamma_2(t) = < t, \sin(\pi t) >$, $t \in [-1,1]$.

1. Sketch each object

f is surface / x=0 → curve, y=0 → curve, or level curves or max/min ..

$\vec{F}$ is a vector field



curl ≠ 0
not conservative

D set in $\mathbb{R}^2$

C is circle (curve w.o. direction)

γ is a space curve with dir, start + finish

Panel 13

Let $f(x,y)=x^2-xy+y^2$, $F(x,y)=<2x-y,2y-x>$, $D=\{(x,y):x^2+y^2\le 1\}$,
$C=\{(x,y):x^2+y^2=1, y\ge 0\}$, $\gamma_1(t)=<t,0>$, $t\in[-1,1]$, and $\gamma_2(t)=<t,\sin(\pi t)>$, $t\in[-1,1]$.

a) $\iint_D f(x,y)dA$

b) $\int_D f(x,y)ds$

$\int_a^b f(x,y)dx$ ok but rare

c) $\int_C f(x,y)ds$

d) $\int_{\gamma_1} f(x,y)dx$

e) $\int_{\gamma_2} f(x,y)dy$

f) $\int_{\gamma_1} f(x,y)dr$

$\int F\,dr = \int M dx + N dy$

g) $\int_{\gamma_1} F(x,y)dx$

h) $\int_D F(x,y)dr$

i) $\int_C F(x,y)dr$

j) $\int_{\gamma_1} F(x,y)dr$

k) $\int_{\gamma_2} F(x,y)dr$

Panel 14

If $\vec{F}$ is a vector field and γ is a curve from A to B parametrized by $r(t)$, $t\in[a,b]$ then

(2D) $\int_\gamma F\,dr = \int_\gamma M(x,y)dx + N(x,y)dy$

(3D) $\int_\gamma F\,dr$ $\int M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$

gives Work

Panel 15

<u>Ex:</u> Find work done by the vector field $\vec{F} = \langle x, y^2, zx \rangle$ along line from $(0,0,0)$ to $(1,2,3)$.

$$W = \int_\gamma \vec{F}\, d\vec{r} = \qquad r(t) = \langle 0,0,0\rangle + t\langle 1,2,3\rangle = \langle t, 2t, 3t\rangle$$

$$= \int_\gamma x\,dx + y^2\,dy + zx\,dz =$$

$$= \int_0^1 t\,dt + (2t)^2\,2dt + (3t)(t)\,3\,dt$$

Panel 16

<u>Fundamental Theorem of Line Integration</u>

Suppose $\vec{F}$ is a conservative vector field. Then

$$\int_C \vec{F}\, d\vec{r} = \int_\gamma (M\,dx + N\,dy + P\,dz)$$

$$= f(B) - f(A)$$

where f is <u>potential of $\vec{F}$</u> and curve γ starts at A and goes to B