Panel 1

Birds-Eye View so far

$f: \mathbb{R} \to \mathbb{R}$ Calc I

$f: \mathbb{R} \to \mathbb{R}^2$ space curves

$f: \mathbb{R}^2 \to \mathbb{R}$ surfaces

$f: \mathbb{R}^2 \to \mathbb{R}^2$

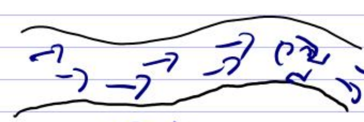
$\mathbb{R}^3 \to \mathbb{R}^3$

Ex: $f(x,y) = \langle x^2+y^2, xy \rangle$

Panel 2

Vector Fields: If for each point P in a region R there is a unique vector having initial point P, then the totality of such vectors is called a vector field.

Ex: Flow of water



2D

Ex: Magnetic Field

2D

Ex: Gravity

3D

Panel 3

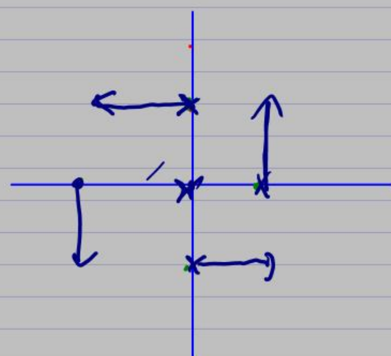
Mathematically, a vector field is given as:

$F(x,y) = \langle M(x,y), N(x,y) \rangle$

$F(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$

Ex: Describe $F(x,y) = \langle -y, x \rangle = -y\vec{i} + x\vec{j}$

(x, y)	F(x, y)
(0,0)	(0,0)
(1,0)	(0,1)
(0,1)	(-1,0)
(-1,0)	(0,-1)



Panel 4

Maple offers "fieldplot" and "fieldplot3d"

```
> with(plots);
> fieldplot([-y, x], x = -2..2, y = -2..2);
> fieldplot3d([-x/(x^2+y^2+z^2)^(3/2), -y/(x^2+y^2+z^2)^(3/2), -z/(x^2+y^2+z^2)^(3/2)], x = -2..2, y = -2..2, z = -2..2);
>
```

Panel 5

Def: Suppose $F(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$

Then

$$\text{curl}(F) = \langle P_y - N_z, -(P_x - M_z), N_x - M_y \rangle$$

$$\text{div}(F) = M_x + N_y + P_z = \nabla \cdot F$$

How to memorize: $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$

$$\text{curl}(F) = (\nabla \times F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle P_y - N_z, -(P_x - M_z), N_x - M_y \rangle$$

$\nabla f = \langle f_x, f_y \rangle$, $f: \mathbb{R}^2 \to \mathbb{R}$

Panel 6

Ex: Let $F(x,y,z) = \langle xy, yz, xz \rangle$. Then

$$\text{curl}(F): \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{vmatrix} = \langle 0 - y, z - 0, 0 - x \rangle$$

$$\text{div}(F) = y + z + x$$

Panel 7

Ex: $F(x,y,z) = \langle xy^2z^4,\ 2x^2y+z,\ y^3z^2 \rangle$

Find curl(F) and div(F)

$\operatorname{div}(F) = y^2z^4 + 2x^2 + 2y^3z$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy^2z^4 & 2x^2y+z & y^3z^2 \end{vmatrix} = \begin{pmatrix} 3y^2z^2 - 1 \\ -(0 - 4xy^2z^3) \\ 4x^2 - 2xyz^4 \end{pmatrix}$$

Panel 8

2. Suppose that $F(x,y,z) = < x^3z, x^2z, xy >$ is some vector field.

a) Find $div(F)$

b) Find $curl(F)$

Panel 9

(?) If $f: \mathbb{R}^3 \to \mathbb{R}$ is a function and $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, which expression is meaningful:

- curl (f) ✗
- grad (f) ✓
- div (F) ✓
- curl (grad (f)) ✓
- grad (F) ✗
- grad (div (F)) ✓
- div (grad (f))
- etc.

div (curl(F)) ✓

curl (~~div~~ (F))

Panel 10

Def: A vector field $\vec{F}$ is conservative if there is $f: \mathbb{R}^3 \to \mathbb{R}$ a function s.t.

$\nabla f = \text{grad}(f) = F$, f is called potential

Ex: Find vector field with potential

$$f(x,y,z) = x^2 - 3y^2 + 4z^2$$

$F = \nabla f = \text{grad}(f) = \langle 2x, -6y, 8z \rangle$

is conservative ✓

Panel 11

Which of the following vector field(s) has as potential function $f(x,y,z) = x^2y^2z^2 + xy + zy$

(a) $\vec{F} = \langle 2x, y, z \rangle$

(b) $\vec{F} = \langle 2xy^2z^2 + x + y \rangle$

(c) $\vec{F} = \langle 2xy^2z^2, y, z \rangle$

(d) $\vec{F} = \langle 2xy^2z^2, x, y \rangle$

(e) $\vec{F} = \langle 2xy^2z^2 + y,\ 2yx^2z^2 + x + z,\ 2zx^2y^2 + y \rangle$

Panel 12

Suppose $\vec{F}$ is conservative, i.e. $F = \langle f_x, f_y, f_z \rangle$

then $\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz},\ -(f_{zx} - f_{xz}),\ f_{yx} - f_{xy} \rangle$ $= 0$, $= 0$, $= 0$

Theorem: If F is a vector field s.t. $\text{curl}(F) = 0$

$\Rightarrow$ F is conservative

Panel 13

If $F = \langle M(x,y), N(x,y), \quad \rangle$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = \langle 0, 0, N_x - M_y \rangle$$

Corollary: $F = \langle M, N \rangle$ is conservative iff $N_x = M_y$!

Panel 14

Which of the following vector fields is not conservative

(a) $F(x,y) = \langle x, y \rangle$ ✓

(b) $F(x,y) = \langle x^2 + y^2, 2xy \rangle$ ✓

(c) $F(x,y) = \langle e^x \cos(y), -e^x \sin(y) \rangle$ ✓

~~(d)~~ $F(x,y) = \langle x^2 \cos(y), -y^2 \sin(x) \rangle$ Not conservative

Panel 15

Find potential function for $\vec{F} = (3 + 2xy, x^2 - 3y^2)$ if exists

$f(x,y) = 3x + x^2y - y^3$

$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} = 2x$ is conservative

there is f s.t. $f_x = M$, $f_y = N$

$f_x = 3 + 2xy \Rightarrow f = \int 3 + 2xy \, dx$

$= 3x + x^2y + C(y)$

$f_y = x^2 + C'(y) = x^2 - 3y^2 \Rightarrow C'(y) = -3y^2$

$\Rightarrow C(y) = -y^3$

Panel 16

Find potential function for $\langle x^2\cos(y), -y^2\sin(x) \rangle$

$M_y = -x^2\sin(y) \neq N_x = -y^2\cos(x)$

No potential function

$f = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$

$\vec{F} = \langle x, y \rangle$

$F(x,y) = \langle y, x \rangle$

$f_x = y \Rightarrow f = yx + C(y)$

$f_y = x + C' = x \Rightarrow f(x,y) = xy + C$

Panel 17

Find potential for $\vec{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$ if exists.

f s.t. $\nabla f = \vec{F}$

$f_x = y^2 \rightarrow f(x,y,z) = xy^2 + C(y,z)$

$f_y = 2xy + C_y = 2xy + e^{3z}, \quad C_y = e^{3z} \Rightarrow C = ye^{3z} + C(z)$

$f = xy^2 + ye^{3z} + C(z)$ c

$f_z = 3ye^{3z} + C' = 3ye^{3z} \rightarrow C' = 0, \; C = \text{const}$