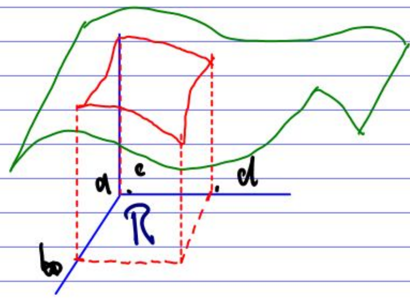
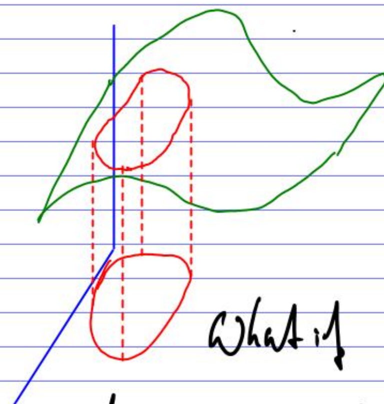
Panel 1

Last Time

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

volume





$R = [a,b] \times [c,d]$

What if domain $\neq$ rectangle

Panel 2

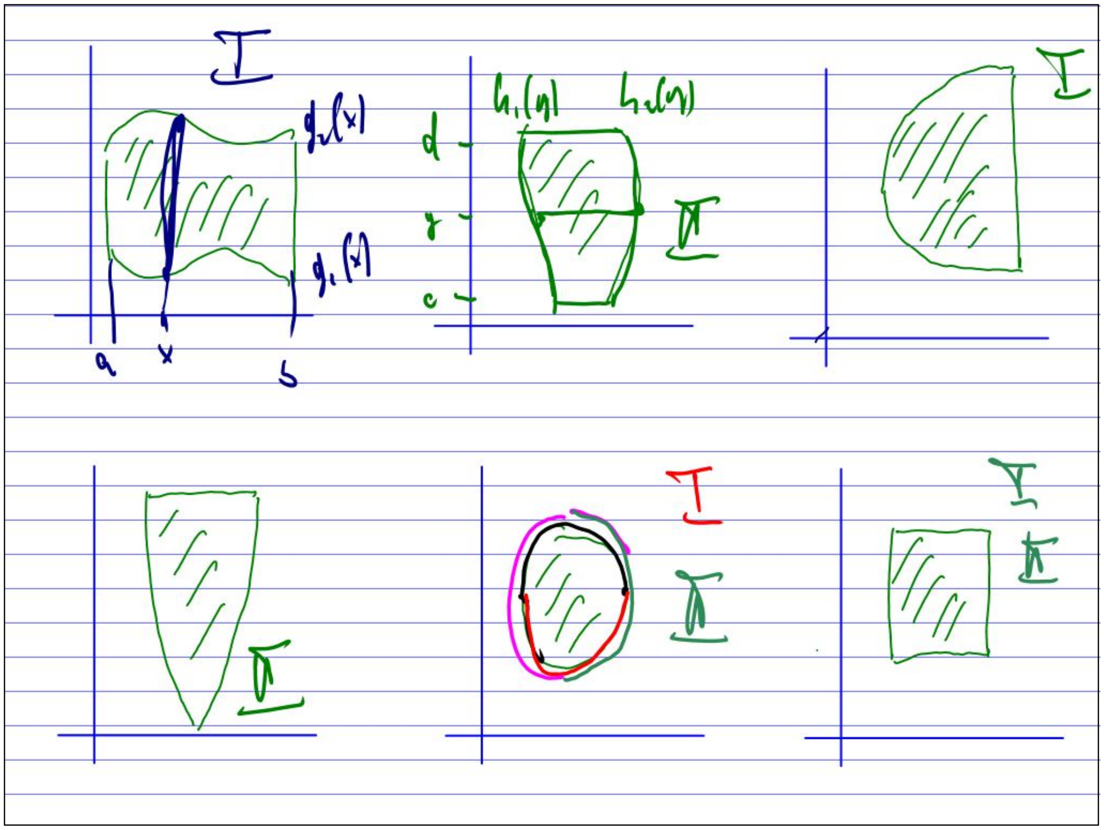
Type 1 Region: $D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Type 2 Region: $D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

Type 1: $$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type 2: $$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Panel 3

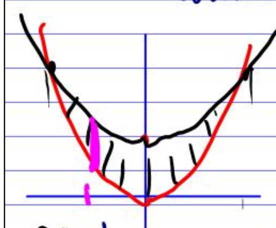


Panel 4

Ex: Find $\iint_D (x+2y)\,dA$ where D is the region bounded by $y = 2x^2$ and $y = 1+x^2$

$2x^2 = 1+x^2$

$x^2 - 1 = 0$



Type I:

$a=-1$ $\quad b=+1$

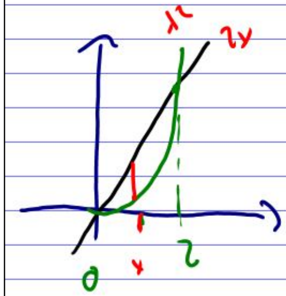
$$\int_{-1}^{1}\left(\int_{2x^2}^{1+x^2} x+2y\,dy\right)dx =$$

$$\int_{-1}^{1}\left(xy + y^2 \Big|_{y=2x^2}^{y=1+x^2}\right) dx = \int_{-1}^{1}\left[x(1+x^2)+(1+x^2)^2\right]-\left[x\,2x^2+(2x^2)^2\right] dx$$

(check!) $= 32/15$

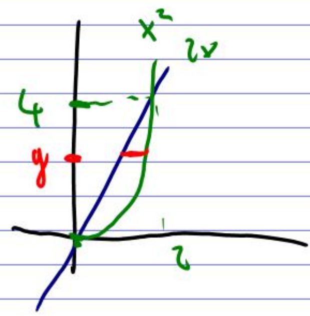
Panel 5

Ex: Volume under $z = x^2+y^2$ above $y = 2x$ and $y = x^2$.



$2x = x^2$, $x = 0, 2$

Type I: $\displaystyle\int_0^2 \int_{x^2}^{2x} x^2+y^2 \, dy \, dx$

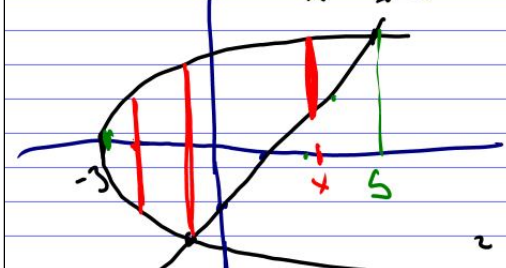


Type 2: $\displaystyle\int_0^4 \int_{y/2}^{\sqrt{y}} x^2+y^2 \, dx \, dy$

Panel 6

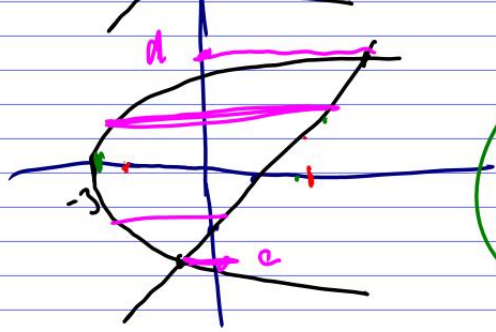
Ex: Find $\iint_D xy \, dA$ where D is bounded by $y = x-1$ and $y^2 = 2x+6$. Should you $\iint xy \, dx\,dy$ or $\iint xy \, dy\,dx$?

$x = y^2 - 6$



$\displaystyle\int_{-3}^{5}\int f(x,y)\, dy \, dx$

Two integrals !!



$\displaystyle\int_c^d\int f(x,y)\, dx \, dy$

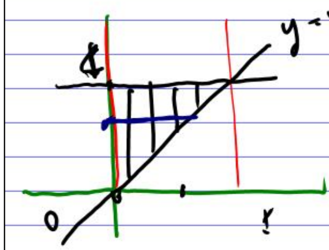
One integral

Panel 7

Ex: Find $\displaystyle\int_{x=0}^{x=1}\int_{x=y}^{1=y} \sin(y^2)\,dy\,dx$ $\quad x\in[0,1],\ y\in[x,1]$

Cannot find $\int \sin(y^2)\,dy$

$$\int_0^1\int_0^y \sin(y^2)\,dx\,dy = \int_0^1 x\sin(y^2)\Big|_{x=0}^{x=y} dy$$



$$= \int_0^1 y\sin(y^2) - 0\,dy = -\tfrac12\cos(y^2)\Big|_0^1 =$$

$$= -\tfrac12\cos(1) + \tfrac12 = \tfrac12\left[1-\cos(1)\right]$$

Panel 8

$$\int_0^2\int_0^1 x^2y\,dx\,dy$$

$$\int_0^1\int_0^{\cancel{y}} \sqrt{1-y^2}\,d\underline{y}\,dx$$

$$\int_0^1\int_0^y \sqrt{1-y^2}\,dx\,dy$$ like :)

$$\int_0^1\int_0^x \sqrt{1-y^2}\,dy\,dx$$ dislike :(

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\,dy\,dx$$

Panel 9

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\, dy\, dx = \frac{1}{2}\cdot\frac{4}{3}\pi(2)^3$$

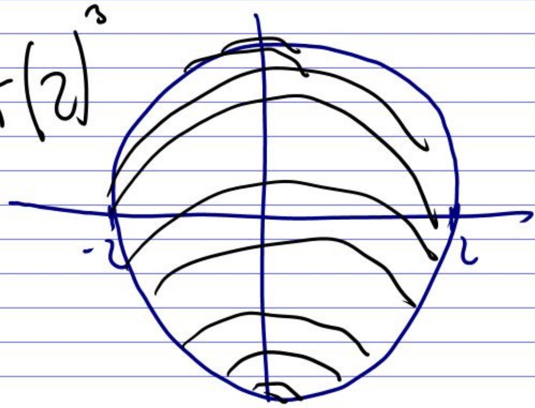
$y = \pm\sqrt{4-x^2}$

$x^2+y^2=4$

integrand:

$z=\sqrt{4-x^2-y^2}$

$x^2+y^2+z^2=4$

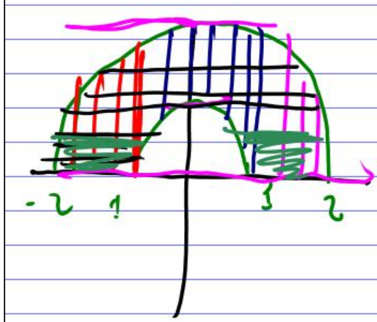


Recall: Volume sphere, radius R: $\frac{4}{3}\pi r^3$

Panel 10

But there are some integral where all tricks (so far) don't work:

$\iint_D (3x+4y^2)\, dA$ where D is region in upper half plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$

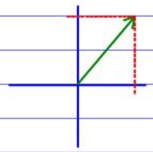


Type 1: 3 integrals $dy\,dx$

Type 2: 3 integrals $dx\,dy$

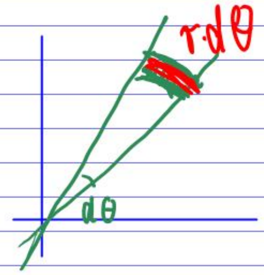
Panel 11

Solution. Polar Coordinates



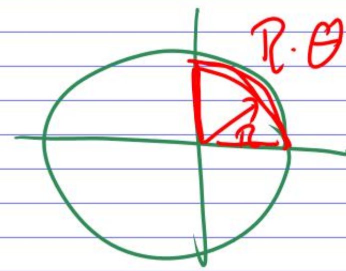
$x = r\cos\theta$

$y = r\sin\theta$



$r\,dr\,d\theta$

$$\iint_R f(x,y)\,dA$$

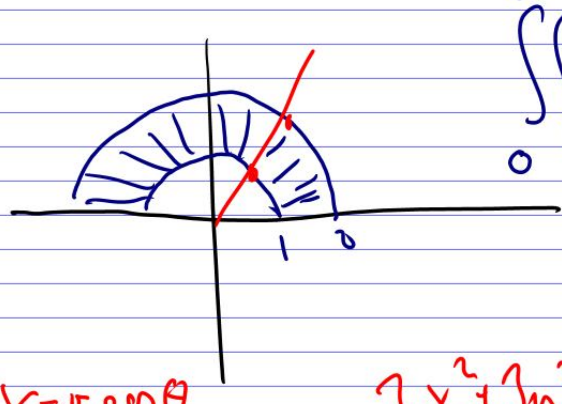


$$dA = dx\,dy = dy\,dx = r\,dr\,d\theta$$

$$(r\,d\theta\,dr)$$

Panel 12

$\iint_D 3x^2+3y^2\,dA$, where D is the region in the upper half plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$



$$\int_0^{\pi}\int_1^{2} 3r^2\, r\,dr\,d\theta = \int_0^{\pi}\int_0^{1} 3r^3\,dr\,d\theta = \frac{3}{4}\pi$$

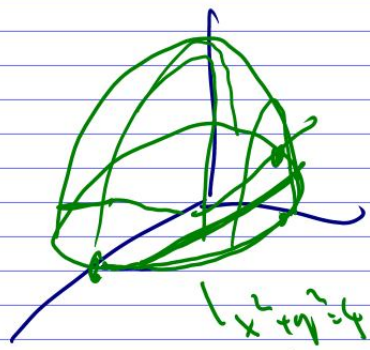
$x = r\cos\theta$

$y = r\sin\theta$

$$3x^2+3y^2 = 3r^2\cos^2(\theta) + 3r^2\sin^2(\theta)$$
$$= 3r^2\, 1$$

Panel 13

Prove that volume of sphere, radius 2, is $\frac{4}{3}\pi(2)^3$



$$V = 2\cdot\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\,dy\,dx$$

$$= 2\cdot\int_{0}^{2\pi}\int_{0}^{2} \sqrt{4-r^2}\,r\,dr\,d\theta$$

$x^2+y^2=4$

$y=\pm\sqrt{4-x^2}$

$z=\sqrt{4-x^2-y^2}$

$$2\cdot\int_{0}^{2\pi} -\frac{1}{2}\,\frac{2}{3}\left(4-r^2\right)^{3/2}\Big|_{r=0}^{r=2} d\theta$$

Panel 14

$$\int_{0}^{2\pi} -\frac{1}{3}\left(4-r^2\right)^{3/2}\Big|_{r=0}^{r=2} d\theta =$$

$$\int_{0}^{2\pi} 0+\frac{1}{3}(4)^{3/2}\,d\theta = 2\cdot\frac{1}{3}\,2^3\cdot 2\pi =$$

$$2\cdot\frac{2}{3}\pi(2)^3$$

$$\frac{4}{3}\pi\, 2^3$$