Panel 1

Max / Min Problems

To find max/min of $z = f(x,y)$:

① Find ∇f, i.e. $\langle f_x, f_y \rangle$

② Solve $f_x = 0$, $f_y = 0$ → critical pts

③ Compute $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ and $D = f_{xx} f_{yy} - (f_{xy})^2$

a) f has min if: $D > 0$, $f_{xx} > 0$

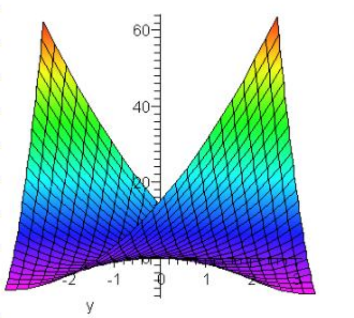
b) f has max if: $D > 0$, $f_{xx} < 0$

c) f has saddle if: $D < 0$

d) no information if: $D = 0$

Panel 2

Suppose $f(x,y) = x^2 + 2y^2 + 4xy$. Find and classify all relative extrema, if any.



$f_x = 2x + 4y = 0$

$f_y = 4y + 4x = 0$

$(x = 0, y = 0)$

$f_{xx} = 2$

$f_{yy} = 4$

$H = \begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix}$

$f_{xy} = 4$

$D = 8 - 16 = -8 < 0$

Thus: $(0,0)$ is a saddle point.

Panel 3

Find and classify critical points for $f(x,y) = 3x - x^3 - 2y^2$

$f_x = 3 - 3x^2 = 0 = 3(1-x)(1+x) \Rightarrow x = +1, -1$

$f_y = -4y = 0 \Rightarrow y = 0$

two critical pts: $(1,0), (-1,0)$

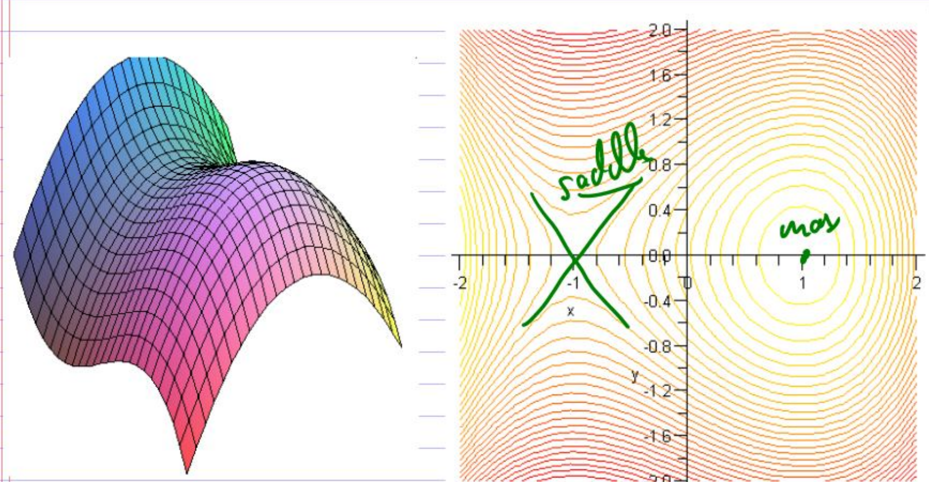
$H = \begin{pmatrix} -6x & 0 \\ 0 & -4 \end{pmatrix}$, $D = \underline{24x}$

at $(1,0)$: $D > 0$, $f_{xx} < 0 \Rightarrow$ max

at $(-1,0)$: $D < 0 \Rightarrow$ saddle point!

Panel 4

$f(x,y) = 3x - x^3 - 2y^2$

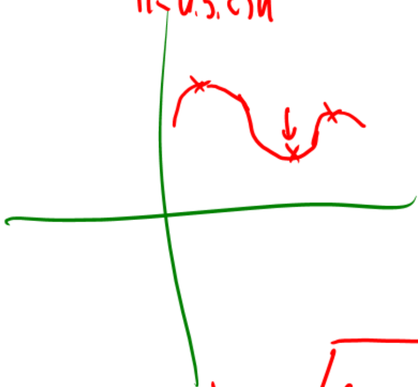


Panel 5

$$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\|\langle a,b,c\rangle\|} = d$$

$$x - y + z = 4$$

$$P(1,2,3)$$



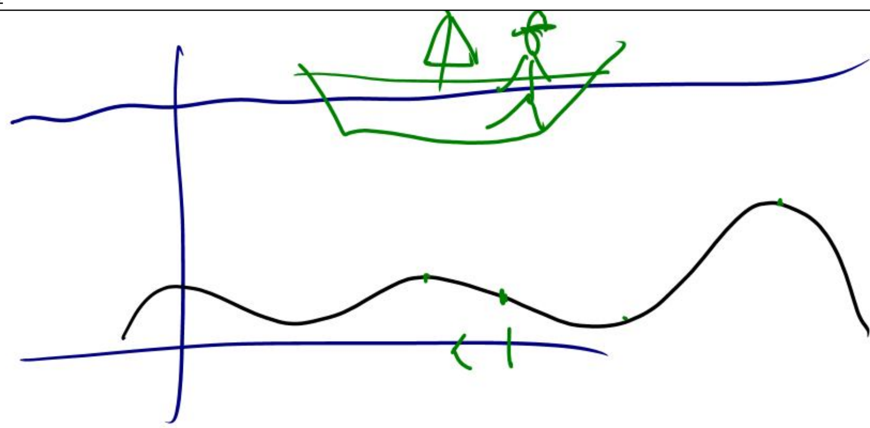
Q lies on plane: $(x_0, y_0, 4 - x_0 + y_0)$

$$\text{dist}(PQ) = \sqrt{(x_0-1)^2 + (y_0-2)^2 + (4 - x_0 + y_0 - 3)^2} = d$$

If d is minimal $\Leftrightarrow$ d^2 is minimal

$\Rightarrow$ Minimize $d^2 = (x-1)^2 + (y-2)^2 + (1-x+y)^2$

Panel 6



Panel 7

Ex: Find and classify the critical points for $f(x,y) = x^3y + 12x^2 - 8y$

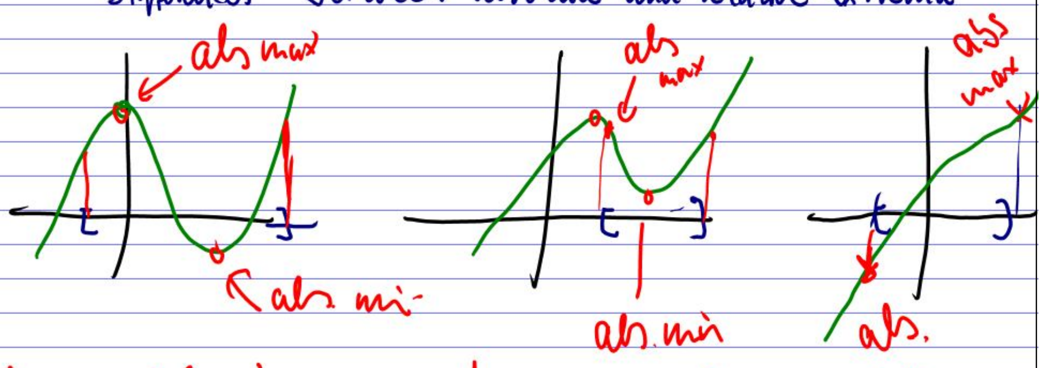
Hint: $\exists$ critical points!

HW

Panel 8

Absolute Max / Min:

Differences between absolute and relative extrema



Thm: $f(x,y)$ is cont on a closed, bounded set D in $\mathbb{R}^2$. Then f has an abs. max and abs. min. They occur only at the critical points or the boundary of D

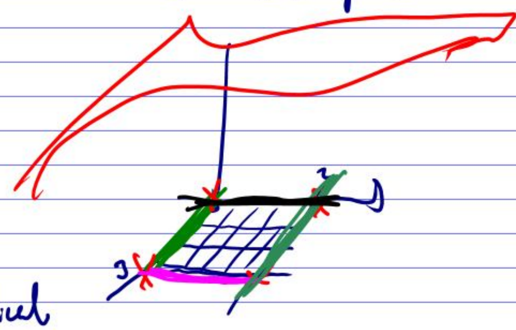
Panel 9

Ex: Find abs. extrema for $f(x,y) = x^2 - 2xy + 2y$ on $[0,3]\times[0,2]$, i.e. $0 \le x \le 3$ and $0 \le y \le 2$

(1) Critical points:

$f_x = 2x - 2y = 0$

$f_y = -2x + 2 = 0$



$x = 1,\ y = 1$ is critical

$y = 0,\ x \in [0,3]$: $f(x) = x^2$, $x = 0$ is critical $(y=0)$

$x = 3,\ y \in [0,2]$: $f(y) = 9 - 6y + 2y = 4y$, no criticals $\quad y = 2$

$y = 2,\ x \in [0,3]$: $f(x) = x^2 - 4x + 4$, $f' = 2x - 4 = 0$, $x = 2$

$x = 0,\ y \in [0,2]$, $f(y) = 2y$ no criticals

Panel 10

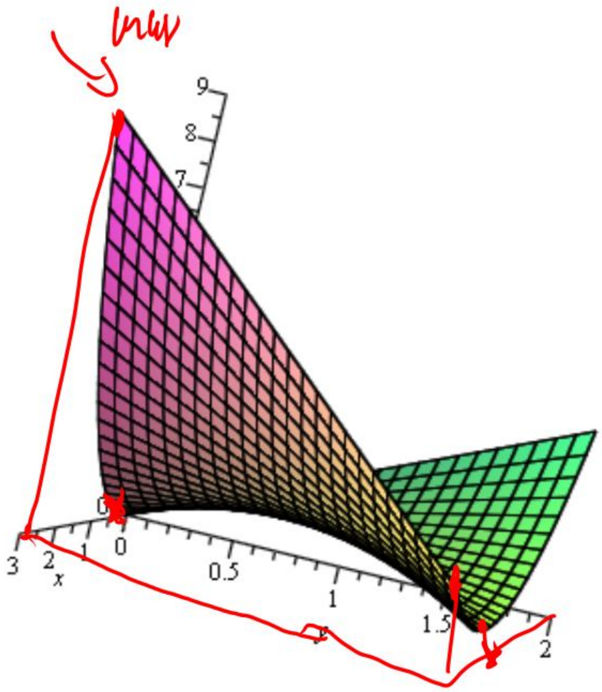
$f(x,y) = x^2 - 2xy + 2y$

(0,0)	0	critical
(2,2)	0	
(1,1)	1	
~~(0,0)~~		end points
(0,2)	4	
(3,0)	9	
(3,2)	2	

abs. min. at (0,0), (2,2)

abs. max at (3,0)

Panel 11



(x,y)	$z=f(x,y)$
$(1,1)$	1
$(2,2)$	0
$(0,0)$	0
$(0,2)$	4
$(3,0)$	9
$(3,2)$	2

Panel 12

$f(x,y)=x^2+2y^2+4xy$ for $(x,y)\in[0,1]\times[0,1]$. → abs. extrema?

① $f_x=0$

$f_y=0$

② $x=0$: $f'=0$

$x=1$: $f'=0$

$y=0$: $f'=0$

$y=1$: $f'=0$

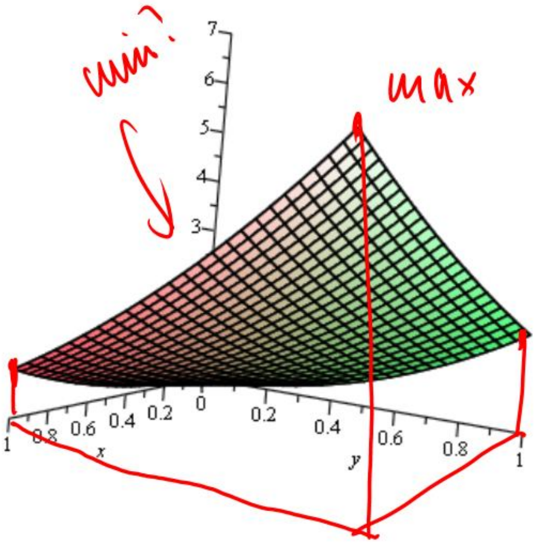
end points

→ ③

	f
crit. pts	
$(0,0)$	
$(0,1)$	
$(1,0)$	
$(1,1)$	

Panel 13

$f(x,y) = x^2 + 2y^2 + 4xy$



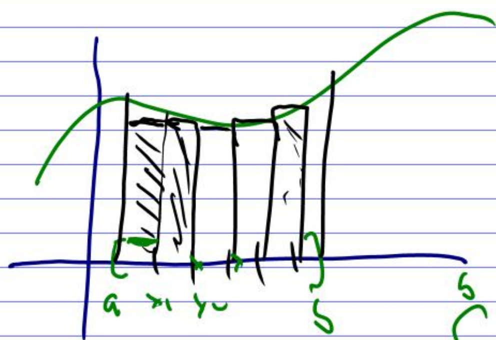
(x,y)	$f(x,y)=z$
$(0,0)$	
$(0,1)$	
$(1,0)$	
$(1,1)$	

Panel 14

Integration: in $\mathbb{R}$.

$f: [a,b] \to \mathbb{R}$

$$\int_a^b f(x)\,dx \stackrel{def}{=} \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x_j$$



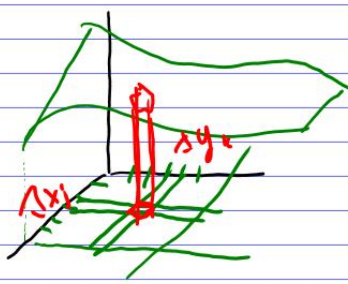
Geometrically:

$\int_a^b f(x)\,dx$ = area under f (if $f > 0$)

Panel 15

Integration in $\mathbb{R}^2$

$$\iint_Q f(x,y)\,dA =$$



$$\lim_{j,k\to\infty} \sum_k \sum_j f(x_j,y_k)\,\Delta x_j\,\Delta y_k$$

Geometrically: $\iint_Q f(x,y)\,dA$ is

Volume under surface $z = f(x,y)$

Panel 16

Fubini's Theorem (How to integrate in $\mathbb{R}^2$)

If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b\int_c^d f(x,y)\,dy\,dx$$

Ex: $\iint_R (x-3y^2)\,dA$, $R=[0,2]\times[1,2]$

(1) $\int_0^2\int_1^2 x-3y^2\,dy\,dx = \int_0^2 \left[xy - y^3 \Big|_{y=1}^{y=2} \right] dx$

$\int_0^2 (2x-8)-(x-1)\,dx = \int_0^2 x+9\,dx = \frac{1}{2}x^2+9x\Big|^2$