Panel 1

**Last Time:** $z = f_x(x-x_0) + f_y(y-y_0) + z_0$

Tangent Plane to $z = f(x,y)$ at $P(x_0, y_0)$

PDE e.g. $\Delta u = 0 \iff \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = u_{xx} + u_{yy} = 0$

Directional deriv. $D_u f = \lim_{h \to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h}$

$u = \langle u_1, u_2 \rangle$ unit vector. $D_u f = \nabla f \cdot u$, $\nabla f = \langle f_x, f_y \rangle$ gradient

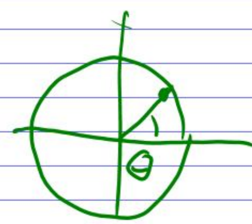
Gradient and its Properties

Panel 2

$f(x,y) = x^2y^3 - y^4$ $P(2,1)$, $\theta = \pi/4$

$\theta = \frac{\pi}{4} \Rightarrow \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \rangle =$

$= \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$




$\nabla f = \langle \underbrace{2xy^3}_{f_x}, \underbrace{3x^2y^2 - 4y^3}_{f_y} \rangle$

$\Rightarrow \nabla f(2,1) = \langle 4, 8 \rangle$

$D_u f(2,1) = \langle 4, 8 \rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \frac{4\sqrt{2} + 8\sqrt{2}}{2} = \frac{12\sqrt{2}}{2} = 6\sqrt{2}$

Means: I stand at $P(2,1)$, $z = 3$

Panel 3

Note: $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

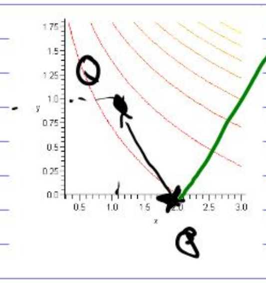
$\Rightarrow \| D_{\vec{u}} f \| = \| \nabla f \cdot u \| = \| \nabla f \| \, \| u \| \cdot \cos(\theta)$

$= \| \nabla f \| \cos(\theta)$

Useful Theorem: The max. value of $|D_{\vec{u}} f|$ is achieved if $\vec{u}$ points in the direction of $\nabla f$ !
The maximum value is $\| \nabla f \|$

Panel 4

Ex: $f(x,y) = xe^y$. Then find rate of change at $P(2,0)$ in direction from P to $Q(1,1)$. In what direction does f have max. rate of change, and what is it?




$\vec{PQ} = \langle -1, 1 \rangle$, $\vec{u} = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle$

$\nabla f = \langle \underbrace{e^y}_{f_x}, \underbrace{xe^y}_{f_y} \rangle$

$\Rightarrow \nabla f(2,0) = \langle 1, 2 \rangle$

$\Rightarrow D_u f = \nabla f \cdot \vec{u} = \langle 1, 2 \rangle \cdot \langle -1, 1 \rangle \frac{1}{\sqrt{2}} = 1/\sqrt{2} \approx 0.7$

$\nabla f$ points to steepest incr: $\langle 1, 2 \rangle$, max. incr. $\| \nabla f \| = \sqrt{5}$

Panel 5

Properties of Gradient

- The gradient is a vector
- Gradient is perp. to level curves
- Gradient points in direction of max. increase
- $\|\nabla f\|$ is the max increase

Ex: Find $\nabla f$ if $f(x,y,z) = \ln(xy^2z^3) = \ln(x) + \ln(y^2) + \ln(z^3)$
$= \ln(x) + 2\ln(y) + 3\ln(z)$

$\left\langle \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \right\rangle$

Find max. rate of change for $f(x,y,z) = \ln(xy^2z^3)$ at $P(1,1,2)$

$\Rightarrow \|\nabla f\| = \left\| \left\langle \frac{1}{1}, \frac{2}{1}, \frac{3}{2} \right\rangle \right\| = \sqrt{1+4+\frac{9}{4}} =$ ____

Panel 6

Ex: Suppose the level curves of an area are given by $f(x,y) = y\ln(x)$. You are standing at $P(1,-3)$ and you are heading in the direction $\langle -4, 3 \rangle$. Are you going up or down? How much? Which way should you go for max change in height?

Panel 7

Name: ______

Quiz

① Consider $f(x,y) = x^2 + 3xy - y^2$. Find

a) $f_x = 2x + 3y$ ✓

b) $\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = (f_x)_y = 3$

$\cancel{(2x+3y)(3x-2y)}$

c) $\nabla f = \langle 2x+3y, 3x-2y \rangle$ ✓

Panel 8

② If $f(x,y) = xy + 3xy^2$. Find tangent plane at $P(1,1,4)$

③ $f(x,y) = x^3 - 3xy + 4y^2$. Find directional derivative in the direction of $\langle 3/5, 4/5 \rangle$

Panel 9

Review of Max/Min problems in $\mathbb{R}$

Find $f'$

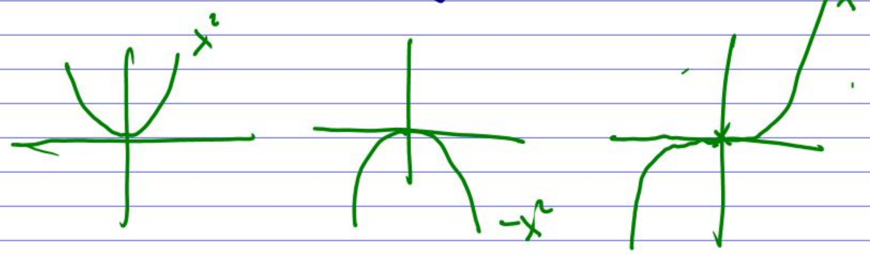
Solve $f'=0 \iff$ critical pts

check: e.g. 2nd deriv. test: check $f''(\underbrace{x_0}_{\text{critical}}) = \begin{cases} >0: \text{min} \\ <0: \text{max} \\ =0: \text{no info} \end{cases}$

1st deriv. test

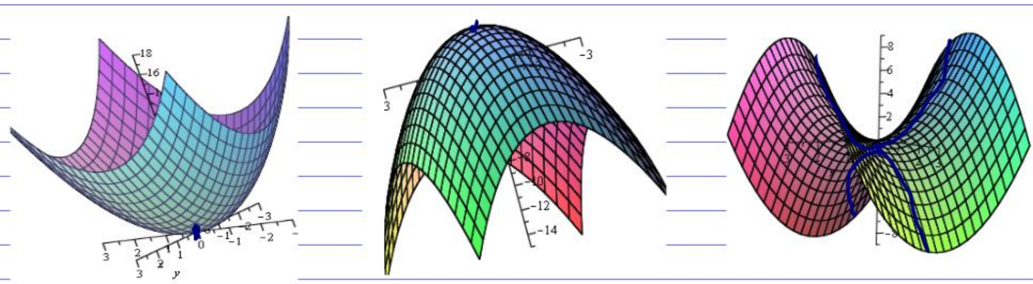
| | | crit. point | |
|---|---|---|---|
| $f'$ | $+$ | | $-$ |
| $f$ | $\nearrow$ | | $\searrow$ |

3 choices

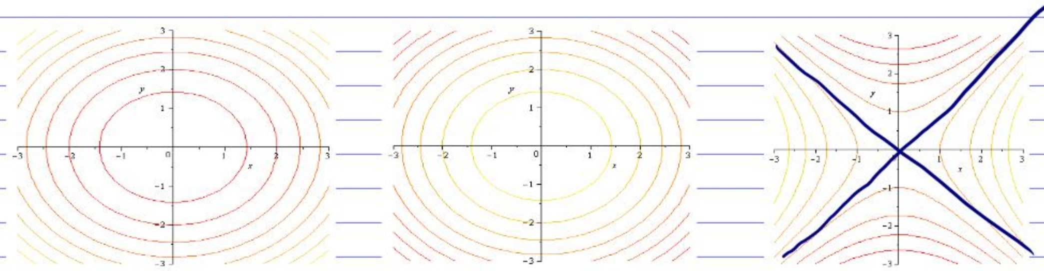



Panel 10

minimum max. saddle point



$f(x,y) = x^2+y^2$ $\qquad f(x,y) = 1-x^2-y^2$ $\qquad f(x,y) = x^2-y^2$



Panel 11

Max / Min Problems

To find max/min of $z = f(x,y)$:

① Find $\nabla f$

② Solve $\nabla f = 0$ (2 equations, 2 unknowns)

③ Compute $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ and $D = f_{xx} f_{yy} - (f_{xy})^2$

a) $f$ has min if: $D > 0$, $f_{xx} > 0$.

b) $f$ has max if. $D > 0$, $f_{xx} < 0$

c) $f$ has saddle if: $D < 0$

d) no information if, $D = 0$

Panel 12

Ex: Find and classify the critical points for

$$f(x,y) = x^2 - 2xy + 3y^2 + 4x$$

1. $f_x = 2x - 2y + 4$, $f_y = -2x + 6y$

2. $2x - 2y + 4 = 0$ $\quad 4y + 4 = 0 \quad y = -1$

$-2x + 6y = 0 \quad x = -3$

3. $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}$ $\quad D = 12 - 4 = 8 > 0$

$f_{xx} > 0$

Panel 13

$f(x,y) = x^2 - 2xy + 3y^2 + 4x$