Panel 1

Is the following function continuous:

$$f(x,y) = \begin{cases} \frac{5x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Yes: $f(0,0) = 0$ exists

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0$$

try this at home! $\delta = \frac{1}{5}\varepsilon$

thus: f is cont at $(0,0)$

Panel 2

Last Time: Partial Derivatives

Def: $f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$

$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$

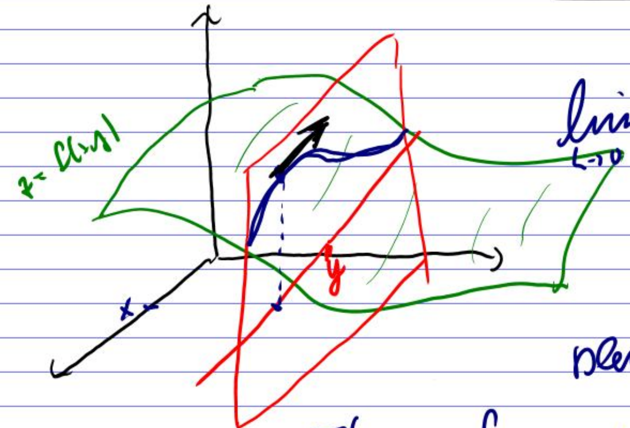
Ex: $f(x,y) = x^2\cos(xy^2)$. Find $f_y(1,1)$:

$f_y = -x^2\sin(xy^2)\cdot 2xy$, $f_y(1,1) = -2\sin(1)$

Higher-Order Derivatives

Panel 3

Geometric Interpretation of Partial Deriv.



$$\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

y fixed is plane parallel to xz

Thus: f_x ~~is slope of tangent of~~ that curve $f(x,y) \cap [y = \text{const}]$

f_x = is slope of "surface" in x-dir.

f_y = is slope of "surface" in y-dir.

Panel 4

Partial derivatives frequently occur in Physics to describe laws of nature as PDEs (partial differential equations). For example: the Laplace PDE

$\Delta u = 0$, Δ is Laplace operator, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\Rightarrow \Delta u = u_{xx} + u_{yy} = 0$

is important in heat conduction and fluid flow.

<u>Ex:</u> Show that $\boxed{f(x,y) = e^x \sin(y)}$ satisfies the above PDE

That means: is $\Delta f = 0$, i.e. $f_{xx} + f_{yy} = 0$?

$f_x = e^x \sin(y)$ $\quad f_{xx} = e^x \sin(y)$

$f_y = e^x \cos(y)$ $\quad f_{yy} = -e^x \sin(y)$ $\quad$ yes

$$0$$

Panel 5

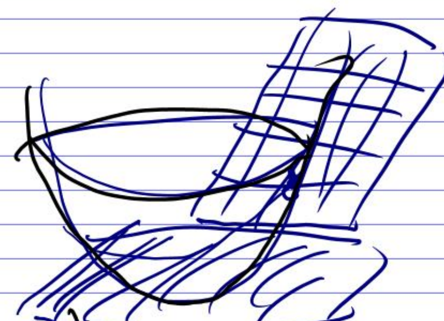
f_x = slope of tangent in x-dir
f_y = slope of tangent in y-dir } form a plane!

Find tangent plane to $z = f(x,y)$

I.e., $ax + by + cz = d$

$z = -\frac{a}{c}x - \frac{b}{c}y + \frac{d}{c}$ i.e.

$z = Ax + By + C$



Want $f_x = \frac{\partial z}{\partial x} = A$

$f_y = \frac{\partial z}{\partial y} = B$

$z = f_x x + f_y y + D = f_x(x - x_0) + f_y(y - y_0) + z_0$

Panel 6

Equation of tangent plane to $f(x,y)$ at (x_0, y_0) is:

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$$

Ex: $f(x,y) = 2x^2 + y^2$. Find tangent plane at $P(1,1,3)$

$f_x = 4x$, $f_x(1,1) = 4$

$f_y = 2y$, $f_y(1,1) = 2$

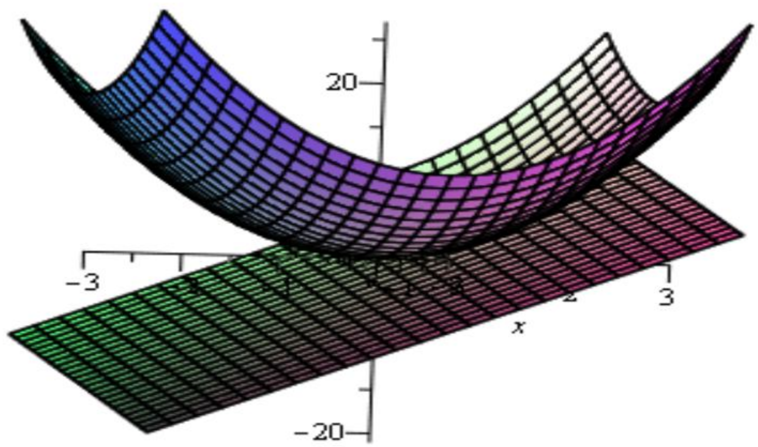
$z = 4(x-1) + 2(y-1) + 3$

Panel 7

$f(x,y) = 2x^2 + y^2$ Tangent plane at $P(1,1,3)$ is:

$z = 4(x-1) + 2(y-1) + 3$

$plot3d\left(\left\{2\cdot x^2 + y^2, 4\cdot(x-1) + 2\cdot(y-1) + 3\right\}, x = -3..3, y = -3..3\right)$



Panel 8

$f(x,y) = x^3y^2 + 5x^2 - 2$

$f(x,y) = x\sin(y) + e^{xy^2}$ → $f_x = \sin(y) + y^2 e^{xy^2}$

$f(x,y) = xe^{xy} + y\ln\frac{x}{y}$ → $f_y = x^2 e^{xy} + \ln\left(\frac{x}{y}\right) + y\cdot\left(\frac{1}{\frac{x}{y}}\right)(-xy^{-2})$

$f(x,y,z) = xy\sin(yz)$

$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

$f(x,y) = \frac{xy}{x^2+y^2}$

$\frac{\partial}{\partial y}\ln\left(\frac{x}{y}\right) = -\frac{1}{y}$

$f(x,y) = \frac{x\sin(x^2+y^2)}{1+y\cos(x)}$

f_{xzy}: $f_x = y\sin(yz)$

$f_{xz} = y^2\cos(yz)$

$f(x,y) = \sqrt{1+(xy)^2}$

2nd order

$f_{xzy} = 2y\cos(yz) - y^2\sin(yz)\cdot z$

Panel 9

The Chain Rule

$f: \mathbb{R} \to \mathbb{R},\ g: \mathbb{R} \to \mathbb{R} \Rightarrow \frac{d}{dx} f(g(x)) =$

Chain Rule in $\mathbb{R}^2$:

$z = f(x,y)\ ,\ x = g(t),\ y = h(t)$

as covered later

$$\Rightarrow \frac{\partial f}{\partial t} =$$

$z = f(x,y)\ ,\ x = g(s,t),\ y = h(s,t)$

$$\Rightarrow \frac{\partial f}{\partial s} =$$

$$\frac{\partial f}{\partial t} =$$

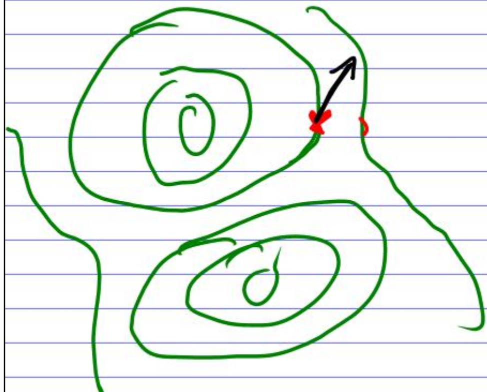
Panel 10

Directional Derivatives:

f_x = slope in x-dir

f_y = slope in y-dir

Slope in arbitrary direction $\vec{v} = \langle v_1, v_2 \rangle$?



Guess: f_x largest

$f_y \approx 0$

Slope in dir $\vec{v} = \langle v_1, v_2 \rangle$

Panel 11

Def: $D_{\vec v}(f(x,y))$ is directional deriv. of f in direction $\vec v$ at point (x,y), where $\vec v$ is a unit vector ($\|\vec v\| = 1$). It is defined as

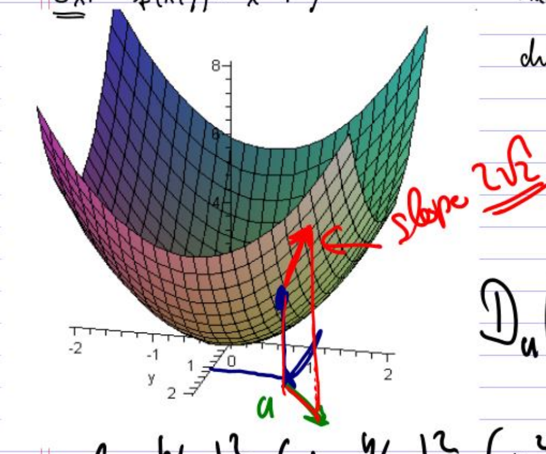
$$D_{\vec v}(f(x,y)) = \lim_{h\to 0} \frac{f(x+hv_1, y+hv_2) - f(x,y)}{h}$$

Note: $D_{(1,0)}(f) = D_{\vec i}(f) = f_x$

$D_{(0,1)} f = D_{\vec j}(f) = f_y$

Panel 12

Ex: $f(x,y) = x^2 + y^2$



Find directional derivative in direction of $\vec u = \frac{1}{\sqrt2}\langle 1,1\rangle$ at $(1,1)$

must have length 1 and

$$D_u(f) = \lim_{h\to 0} \frac{f(x+h/\sqrt2,\ y+h/\sqrt2) -}{h}$$

$$\lim_{h\to 0} \frac{(x+h/\sqrt2)^2 + (y+h/\sqrt2)^2 - (x^2+y^2)}{h} = \frac{\cancel{x^2} + 2x\frac{h}{\sqrt2} + \frac{h^2}{2} + \cancel{y^2} + 2y\frac{h}{\sqrt2} + \frac{h^2}{2} - \cancel{x^2} - \cancel{y^2}}{h}$$

Panel 13

$$\lim \frac{\cancel{x^2} + 2x\frac{h}{\sqrt{2}} + \frac{h^2}{2} + \cancel{y^2} + 2y\frac{h}{\sqrt{2}} + \frac{h^2}{2} - \cancel{x^2} - \cancel{y^2}}{h} =$$

$$\lim \frac{\cancel{h}(\sqrt{2}x + \sqrt{2}y + h)}{\cancel{h}} = \underline{\underline{\sqrt{2}x + \sqrt{2}y}}$$

$$D_{\vec{u}} f(1,1) = \underline{\underline{2\sqrt{2}}}$$

Panel 14

<u>Thm:</u> Suppose $\vec{u} = \langle a, b \rangle$ is a unit vector. Then

$$D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$$

<u>Ex:</u> $f(x,y) = x^2 + y^2$, $\vec{u} = \frac{1}{\sqrt{2}}\langle 1, 1 \rangle$. Find $D_{\vec{u}} f$ at $(1,1)$

$f_x = 2x$

$f_y = 2y$

$$D_u f(x,y) = 2x \frac{1}{\sqrt{2}} + 2y \frac{1}{\sqrt{2}}$$

$$= \underline{\underline{\sqrt{2}x + \sqrt{2}y}}$$

Panel 15

Ex: $f(x,y) = x^3 - 3xy + 4y^2$. Find directional derivative in the direction of $\langle \cos(\pi/6), \sin(\pi/6) \rangle = \vec{u}$

$\cos(\pi/6) = \sqrt{3}/2$, $\sin(\pi/6) = 1/2$

$\|u\| = \sqrt{\cos^2(\frac{\pi}{6}) + \sin^2(\frac{\pi}{6})} = 1$

$\langle f_x, f_y \rangle = \langle 3x^2 - 3y, -3x + 8y \rangle$

$D_u f = \langle 3x^2 - 3y, -3x + 8y \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle =$

$= \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y)$

Panel 16

Gradient: If $f(x,y)$ is a function of 2 variables (or 3) then the gradient of f is:

$\text{grad}(f) = \nabla f = \langle f_x, f_y \rangle$, $\langle f_x, f_y, f_z \rangle$

is a vector with components f_x and f_y

Note: $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$df = f_x\,dx + f_y\,dy$ ← total differential!

Ex: $f(x,y,z) = x\sin(yz)$. find ∇f

$\langle \underbrace{\sin(yz)}_{f_x}, \underbrace{xz\cos(yz)}_{f_y}, \underbrace{xy\cos(yz)}_{f_z} \rangle$