Tangents, Normals, BiNormals

We are looking at space curves

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

We already know how to compute:

length $\quad s = \int_a^b \|r'(t)\|\, dt$

tangent: $\quad r'(t) = \langle x'(t), y'(t), z'(t) \rangle$

unit tangent $\quad T(t) = \dfrac{1}{\|r'(t)\|}\, r'(t)$

curvature: $\quad \kappa = \dfrac{\|T'(t)\|}{\|r'(t)\|} = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$

The first formula for curvature involves the derivative of $T(t)$, almost always difficult! Thus, the second formula is usually preferred.

Ex: Find κ for $r(t) = \langle t, t^2 \rangle$

We need curve in $\mathbb{R}^3$ so: $r(t) = \langle t, t^2, 0 \rangle$

$$\Rightarrow r'(t) = \langle 1, 2t, 0 \rangle, \quad r''(t) = \langle 0, 2, 0 \rangle \Rightarrow r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 4 \rangle$$

Thus: $\kappa = \dfrac{\|r'\times r''\|}{\|r'\|^3} = \dfrac{\|\langle 0,0,4\rangle\|}{\|\langle 1,2t,0\rangle\|^3} = \dfrac{4}{(1+4t^2)^{3/2}}$

In some other cases either formula works.

Next we want to find vectors perpendicular to T, based on the following fact:

Theorem: If $\vec{r}(t)$ is a space curve s.t.

$\|r(t)\| = \text{constant}$ then

$\vec{r}\,'(t)\cdot\vec{r}(t) = 0$, i.e. $\vec{r}\,'$ is perp. to $\vec{r}$

Proof: $\|r(t)\| = \text{const} \Rightarrow \|r(t)\|^2 = \text{const}$

$\Rightarrow \vec{r}(t)\cdot\vec{r}(t) = c$ because $r\cdot r = \|r\|^2$

Differentiating both sides gives

$\frac{d}{dt}\,\vec{r}(t)\cdot\vec{r}(t) = r'(t)\cdot r(t) + r(t)\cdot r'(t) = \frac{d}{dt}\,c = 0$

Thus: $2\,r'(t)\cdot r(t) = 0$ or $r'(t)\cdot r(t) = 0$

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Def: For a smooth curve $\vec{r}(t)$ with unit tangent $\vec{T}(t)$, we define the unit normal vector $\vec{N}(t)$ as

$N(t) = \dfrac{1}{\|T'(t)\|}\,T'(t)$

Note: because $\|T(t)\| = 1$ and prev. theorem we know that $\vec{N}$ and $\vec{T}$ are perpendicular.

Finally, we define the binormal vector:

Def: If $\vec{r}(t)$ is a smooth curve with unit tangent $\vec{T}$ and unit normal $\vec{N}$ we define the binormal vector $\vec{B}(t)$ as:

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Note: Because of the properties of the cross product, $\vec{B}$ is perp. to $\vec{T}$ and $\vec{N}$. Those three vectors form a "local coordinate system" at any point of $\vec{r}(t)$. Time for examples:

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find T, N, and B.

First: $r'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$$\|r'(t)\| = \sqrt{2}$$

$$\Rightarrow T(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$$

Next: $T'(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), -\cos(t), 0 \rangle$

$$\|T'(t)\| = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow B(t) = \frac{1}{\|T'\|} T' = \sqrt{2} \cdot \frac{1}{\sqrt{2}} \langle -\sin(t), -\cos(t), 0 \rangle$$

$$\begin{array}{cccc} & i & j & k \\ \text{So } T(t) = \frac{1}{\sqrt{2}} & \langle -\sin(t), & \cos(t), & 1 \rangle \end{array}$$

$$N(t) = \underline{\underline{\langle -\cos(t), -\sin(t), 0 \rangle}}$$

$$\Rightarrow B = T \times N = \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), \sin^2(t) + \cos^2(t) \rangle$$

$$= \underline{\underline{\frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle}}$$

Note: You can check that

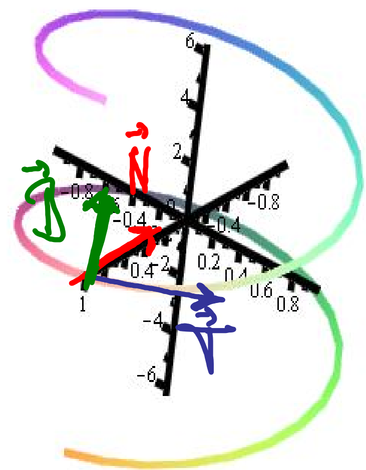
$T \cdot N = 0$, $T \cdot B = 0$, $N \cdot B = 0$ and

$\|T\| = 1$, $\|N\| = 1$, $\|B\| = 1$

<u>Ex:</u> For $r(t) = \langle \cos(t), \sin(t), t \rangle$ find and draw the vectors T, N, B for $t = 0$.

We have already computed the vectors above, so for $t = 0$: $T = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$,

$N = \langle -1, 0, 0 \rangle$ and $B = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle$. They fit onto $r(t)$ as follows:



you can see how these vectors form a local co-ordinate system at $\langle 1, 0, 0 \rangle$

The previous example worked out nicely because $\|T'\|$ was constant. We are not always that lucky.

Ex: Take $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$. Find T, N, B, and κ.

First we need $r'(t) = \langle 2t, 2t^2, 1 \rangle$

and $\|r'(t)\| = \sqrt{1 + 4t^2 + 4t^4} = \sqrt{(1+2t^2)^2} = 1 + 2t^2$

$\Rightarrow$ $\underline{T(t) = \frac{1}{1+2t^2} \langle 2t, 2t^2, 1 \rangle}$

Next we multiply the scalar in:

$$T(t) = \left\langle \frac{2t}{1+2t^2}, \frac{2t^2}{1+2t^2}, \frac{1}{1+2t^2} \right\rangle$$

We now need T', unfortunately:

$$\underline{T'(t)} = \left\langle \frac{2(1+2t^2) - 2t\cdot 4t}{(1+2t^2)^2}, \frac{4t(1+2t^2) - 2t^2\cdot 4t}{(1+2t^2)^2}, -\frac{4t}{(1+2t^2)^2} \right\rangle =$$

$$= \frac{1}{(1+2t^2)^2}\left\langle 2-4t^2, 4t, -4t \right\rangle =$$

$$= \underline{\underline{2/(1+2t^2)^2 \left\langle 1-2t^2, 2t, -2t \right\rangle}} \text{ and}$$

$$\underline{\|T'\|} = \frac{2}{(1+2t^2)^2}\sqrt{(1-2t^2)^2 + 4t^2 + 4t^2} =$$

$$= \frac{2}{(1+2t^2)^2}\sqrt{1-\cancel{4t^2}+4t^4 \cancel{+4t^2}+4t^2} =$$

$$= \frac{2}{(1+2t^2)^2}\sqrt{(1+2t^2)^2} = \underline{\underline{\frac{2}{(1+2t^2)}}}$$

Thus:

$$\underline{N} = \frac{\cancel{2}}{(1+2t^2)^{\cancel{2}}}\left\langle 1-2t^2, 2t, -2t \right\rangle \Big/ \cancel{2}/\cancel{(1+2t^2)} =$$

$$= \underline{\underline{1/(1+2t^2) \left\langle 1-2t^2, 2t, -2t \right\rangle}}$$

So far we have:

$r'(t) = \langle 2t, 2t^2, 1 \rangle$

$\|r'(t)\| = 1 + 2t^2$

$T(t) = \frac{1}{1+2t^2} \langle 2t, 2t^2, 1 \rangle$

$N(t) = \frac{1}{1+2t^2} \langle 1-2t^2, 2t, -2t \rangle$

To find B, I find the cross product of T and N, ignoring the length for now:

$$\begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{vmatrix} 2t & 2t^2 & 1 \\ 1-2t^2 & 2t & -2t \end{vmatrix} \end{array} = \langle -4t^3 - 2t, -[-4t^2 - (1-2t^2)], 4t^2 - 2t^2(1-2t^2) \rangle$$

$$= \langle -4t^3 - 2t, 2t^2+1, 4t^4 + 2t^2 \rangle =$$

$$= \langle -2t(2t^2+1), 2t^2+1, 2t^2(2t^2+1) \rangle =$$

$$= (2t^2+1) \langle -2t, 1, 2t^2 \rangle$$

So $\vec{B}$ has direction $\langle -2t, 1, 2t^2 \rangle$ and must have length 1 so that

$B(t) = \frac{1}{1+2t^2} \langle -2t, 1, 2t^2 \rangle$

You can check that $T \cdot N = T \cdot B = N \cdot B = 0$

That leaves us to find $\kappa = \dfrac{\|r' \times r''\|}{\|r'\|^3}$

$$\begin{array}{cccc} & i & j & k \\ r'(t) = \langle & 2t, & 2t^2, & 1 \rangle \end{array}$$

$r''(t) = \langle 2, 4t, 0 \rangle$

$\Rightarrow r' \times r'' = \langle 4t, +2, 8t^2 - 4t^2 \rangle =$

$= \langle 4t, 2, 4t^2 \rangle$ so that

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\sqrt{16t^2 + 4 + 16t^4}}{(1+2t^2)^3} =$$

$$= \frac{2\cancel{\sqrt{(1+2t^2)^2}}}{(1+2t^2)^{\cancel{3}\,2}} = \frac{2}{(1+2t^2)^2}$$

Woah, quite a lot of work. But there is no shortcut, so buckle up and do it!

This is the last topic we'll need for our exam.