Panel 1

Last Time

Intersections of (a) lines, (b) planes, (c) plane + line

Distances of

$P(x_0, y_0)$ and line $ax+by+c=0$ in $\mathbb{R}^2$

$P(x_0, y_0, z_0)$ and plane $ax+by+cz+d$ in $\mathbb{R}^3$

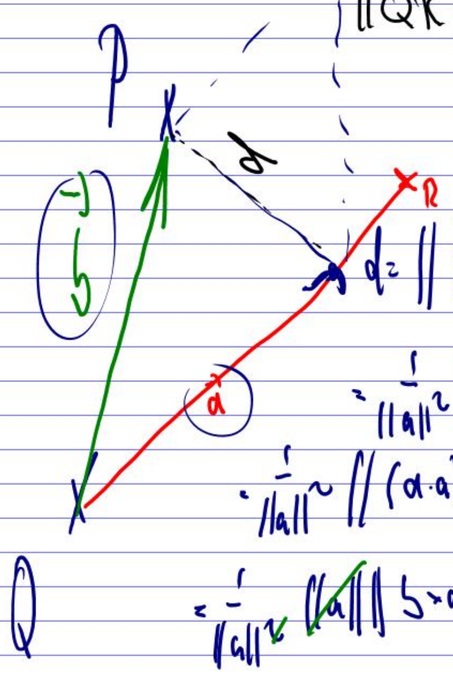
$P(x_0, y_0, z_0)$ and line through Q, R in $\mathbb{R}^3$

see handout

Panel 2

Show that $d = \dfrac{\|QR \times QP\|}{\|QR\|}$ Hint: $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$

dist. between P and line through QR



$$d = \left\| \vec{b} - \text{proj}_{\vec{a}} \vec{b} \right\| = \left\| \vec{b} - \frac{a \cdot b}{\|a\|} \vec{a} \right\| =$$

$$= \frac{1}{\|a\|^2} \left\| \|a\|^2 \vec{b} - (a \cdot b) \vec{a} \right\| =$$

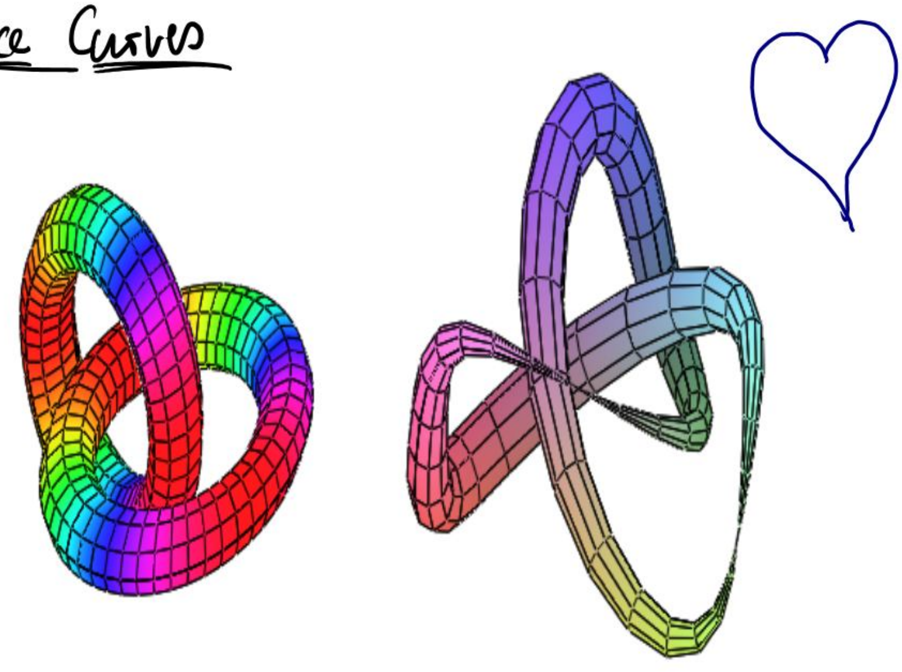
$$= \frac{1}{\|a\|^2} \left\| (a \cdot a) b - (a \cdot b) a \right\| \underset{\text{h.w.}}{=} \frac{1}{\|a\|^2} \left\| a \times (b \times a) \right\| =$$

$90°$

$$= \frac{1}{\|a\|^{\cancel{2}}} \cancel{\|a\|} \|b \times a\| \cdot \sin(\theta) = \frac{\|a \times b\|}{\|a\|}$$

Panel 3

Space Curves



Panel 4

Space Curves

Def: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a vector-valued function with component functions f, g, and h [$x(t)$, $y(t)$, $z(t)$]

Many concepts work as they should: If $\vec{r}(t)$ is vector-valued function then

Limit: $\lim\limits_{t \to t_0} \vec{r}(t) = \langle \lim\limits_{t \to t_0} f(t), \lim\limits_{t \to t_0} g(t), \lim\limits_{t \to t_0} h(t) \rangle$

Derivative: $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Integral: $\int \vec{r}(t)\,dt = \langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \rangle$

Panel 5

Ex: $r(t) = \left\langle \frac{t^2-2t}{t}, \frac{\cos(t)-1}{t}, \frac{\sin(t)}{t} \right\rangle$

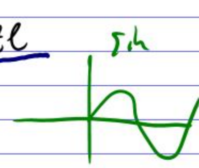
Find $\lim\limits_{t\to 0} r(t) = \langle -2, 0, 1 \rangle$

$r(t) = \left\langle t^3 + \frac{1}{\sqrt{t}} + \pi^2, t\cdot \ln(t), \frac{e^t\cdot\cos(t)}{\sin(t)} \right\rangle$

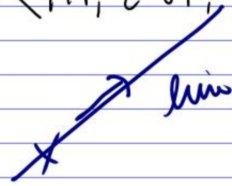
Find $r'(t) = \left\langle 3t^2 - \frac{1}{2}t^{-3/2}, \underbrace{\ln(t)+t\tfrac{1}{t}}_{\ln(t)+1}, \frac{(\quad)-(\quad)}{\sin^2(t)} \right\rangle$

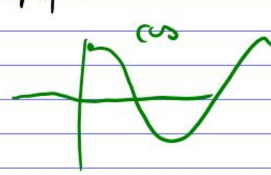
Panel 6

The problem: with vector-valued functions is to visualize them, and interprete the deriv. + integrals:



Ex: $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$ - describe graph





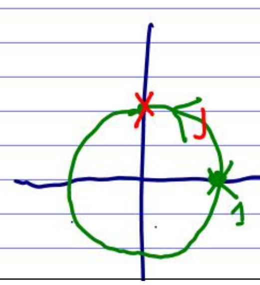
Ex: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ - describe graph

spiral (slinky) around z-axis

$x^2+y^2=1$

$t\in[0,2\pi]$: $\langle \cos(t), \sin(t) \rangle$

$t\in[0,2\pi]$ $\langle \sin(t), \cos(t) \rangle$



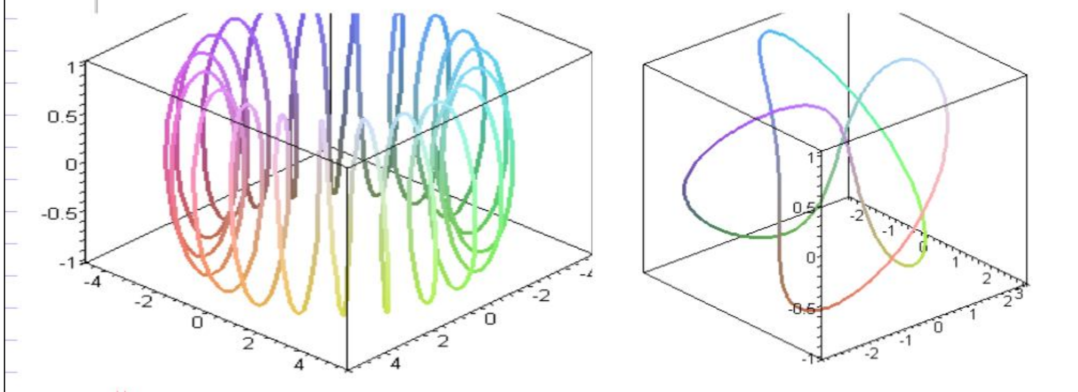
Panel 7

Sketch graph of

$r_1(t) = \langle (4+\sin(20t))\cos(t),\ (4+\sin(20t))\sin(t),\ \cos(20t) \rangle$

$r_2(t) = \langle (2+\cos(1.5t))\cos t,\ (2+\cos(1.5t))\sin(t)\ \ \sin(1.5t) \rangle$

```
> with(plots);
> spacecurve( [(4+sin(20*t))*cos(t), (4+sin(20*t))*sin(t), cos(20*t)], t=0..2*Pi, numpoints=500);
> spacecurve( [(2+cos(1.5*t))*cos(t), (2+cos(1.5*t))*sin(t), sin(1.5*t)], t=0..4*Pi, numpoints=500);
> spacecurve([t, t^2, t^3], t=0..2);
```

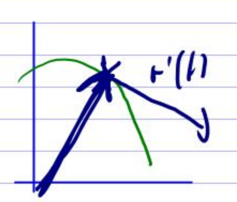


Panel 8

Derivatives of Space Curves aka Vector-valued functions

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ and f, g, h are differentiable

then $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$



$\underline{Ex}$: $\vec{r}(t) = \langle 1+t^3, te^{-t}, \sin(2t) \rangle$

Find $\vec{r}(0)$ and $\vec{r}'(0)$

Compute $\vec{r}(0) \cdot \vec{r}'(0)$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$\vec{r}'(t) = \langle 3t^2, e^{-t} - te^{-t}, 2\cos(2t) \rangle$

$r'(0) = \langle 0, 1, 2 \rangle$

(r' is velocity vector; $\|r'\|$ is speed)

Panel 9

Ex. Find equation of ~~tangent~~ line to $r(t) = \langle 2\cos t, \sin t, t \rangle$ at the point $P(0, 1, \pi/2)$

$\ell(t) = P + t\vec{v}$

$\vec{v} = \vec{r}\,'(t = \frac{\pi}{2}) = \langle -2, 0, 1 \rangle$

$r'(t) = \langle -2\sin(t), \cos(t), 1 \rangle$

t s.t. $r(t) = \langle 2\cos(t), \sin(t), t \rangle = \langle 0, 1, \frac{\pi}{2} \rangle$

$\Rightarrow t = \frac{\pi}{2}$

$\ell(t) = (0, 1, \frac{\pi}{2}) + t\langle -2, 0, 1 \rangle$

Panel 10

Proof: $\frac{d}{dt}\left[\vec{u}(t) \overset{\text{dot}}{\cdot} \vec{v}(t)\right] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$

$\mathbb{R}^2$: $u(t) = \langle f_1(t), g_1(t) \rangle$

$v(t) = \langle f_2(t), g_2(t) \rangle$

$(u \cdot v) = f_1(t) \cdot f_2(t) + g_1(t)\, g_2(t) \quad \Big| \frac{d}{dt}$

work, factor, hope, ...

HW

#

Panel 11

Integrals of Space Curves aka vector valued functions

If $r(t) = \langle f(t), g(t), h(t) \rangle$ and f, g, h are integrable

then $\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$

Ex: If $\vec{r}(t) = 2\cos(t)\,\vec{i} + \sin(t)\,\vec{j} + 2t\,\vec{k}$, find $\int_0^{\pi/2} \vec{r}(t)\,dt$

$r(t) = \langle 2\cos(t), \sin(t), 2t \rangle$

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \left\langle 2\int_0^{\pi/2} \cos(t)\,dt, \int_0^{\pi/2} \sin(t)\,dt, \int_0^{\pi/2} 2t\,dt \right\rangle$$

$$= \left\langle 2, 1, \frac{\pi^2}{4} \right\rangle$$

Panel 12

Arc Length (

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ then $L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h')^2}\,dt$

$$L = \int_a^b \sqrt{(f')^2 + (g')^2 + (h')^2}\,dt = \int_a^b \|r'(t)\|\,dt$$

Ex: Find length of $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$, $t = 0$ to 2π

$$L = \int_0^{2\pi} \|r'\|\,dt = \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\,dt = \int_0^{2\pi} 1\,dt = 2\pi$$

length of unit circle!!

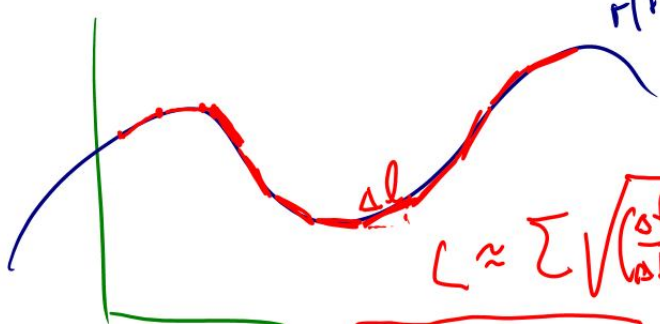
Panel 13

$$\Delta l^2 = \Delta x^2 + \Delta y^2$$

$$L \approx \sum \Delta l = \sum \sqrt{\Delta x^2 + \Delta y^2}$$

$$r(t) = \langle f(t), g(t) \rangle$$

$$\sum \Delta t \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$



$$L \approx \sum \sqrt{\left(\frac{\Delta f}{\Delta t}\right)^2 + \left(\frac{\Delta g}{\Delta t}\right)^2}\,\Delta t$$

$$L = \lim_{\Delta t \to 0} \sum \sqrt{\left(\frac{\Delta f}{\Delta t}\right)^2 + \left(\frac{\Delta g}{\Delta t}\right)^2}\,\Delta t = \int_a^b \sqrt{(f')^2 + (g')^2}\,dt$$

Panel 14

Ex: Find length of $r(t) = \langle \cos(t), \sin(t), t \rangle$ from $(1,0,0)$ to $(1,0,2\pi)$.

HW