Lines and Planes Review

Line in $\boldsymbol{R^n}$: $l(t) = P + t\vec{v} = < p_1, p_2, p_3 > +t < v_1, v_2, v_3 > =< p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 >$
through the point $P =< p_1, p_2, p_3 >$ in the direction of $\vec{v} =< v_1, v_2, v_3 >$

Plane in $\boldsymbol{R^3}$: $ax + by + cz + d = 0$ with normal vector $< a, b, c >$

Intersection of:

a) Two lines $l_1(t) = P + t\vec{v} =< p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 >$ and
$l_2(t) = Q + t\vec{w} =< q_1 + tw_1, q_2 + tw_2, q_3 + tw_3 >$:
Set $l_1(t) = l_2(s)$ and solve over-determined system for *t* and *s* if possible

b) A line $l(t) = P + t\vec{v} =< p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 >$ and a plane $ax + by + cz + d = 0$:
substitute *x*, *y*, and *z* component of the line into the equation of the plane, solve for *t*

c) Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$:
directional vector of the line of intersection must be perpendicular to both planes. Thus $v = n_1 \times n_2$. To find the point *P*, arbitrarily set x = 0 (or y = 0 or z = 0) so that the two equations for the planes reduce to a system of two variables. Solve that for the remaining variables to find the point P

Distances between:

a) A line $ax + by + c = 0$ and a point $P(x_0, y_0)$ in R^2:
The distance is $d = \frac{|ax_0 + b_0 + c|}{||<a,b>||}$

b) A plane $ax + by + cz + d = 0$ and a point $P(x_0, y_0, z_0)$ in R^3:
Find any point Q on the plane, then $d = proj_n(PQ) = \frac{|PQ \cdot \vec{n}|}{||\vec{n}||} = \frac{|ax_0 + by_0 + cz_0 + d|}{||\vec{n}||}$

c) A line though the points Q and R in R^3 and a point $P(x_0, y_0, z_0)$:
The distance is $d = \frac{\left|| \overrightarrow{QR} \times \overrightarrow{QP} ||\right|}{||\overrightarrow{QR}||}$ (see picture)

d) A plane and a line:
if the are not parallel, they intersect so distance is zero. If they are parallel, take any point P on the line and use formula (b)

e) Two planes:
if they are not parallel, they intersect so distance is zero. If they are parallel, take any point P on the first plane and use formula (b)

f) Two lines:
if they are not parallel, they likely won't intersect but if they do, the distance between them would clearly be zero. If they don't intersect, then the distance between the two lines will be perpendicular to both lines. Thus, find a point P on the first line, a point Q on the second line, and project that to the cross product of the two directional vectors.

Exercises

1. True/False

   a) If two lines in R^2 are not parallel, they must intersect

   b) If two lines in R^3 are not parallel, they must intersect

   c) If a line and a planes are not parallel, they must intersect

   d) If two planes are not parallel, they must intersect

   e) The planes $2x - 3y + z = 1$ and $3x + 2y + z - 1 = 0$ are parallel

   f) The lines $l_1(t) =< 1 + t, 2 - t, 3 + 2t >$ and $l_2(t) =< 3,0,1 > +t < 1, -1, -1 >$ are parallel

   g) The plane $x - 3y + 2z - 9 = 0$ and the line $l(t) =< 0,0,1 > +t < -2,6,-4 >$ are parallel

   h) The point $P(3,4,1)$ is on the line $l(t) =< 3,3,3 > +t < 0,1,-2 >$

   i) The point P(3,4,1) is on the plane $x - 5y + 10z + 7 = 0$

   j) The line $l(t) =< 3 - t, 2 + t, 1 + 3t >$ is contained in the plane $2x - y + z = 10$

2. Compute the intersection between the given objects if possible. If that's not possible, state why:

   a) Between $< 2t - 2, 3 - 2t, t >$ and $< t + 1, 2 - 3t, 2t >$

   b) Between $< 2t - 2, 3 - 2t, t >$ and $x + y + z = 1$

   c) Between $x + y + z = 1$ and $3x - 2y - z = 0$

3. Compute the distance between the following objects if possible. If that's not possible, state why.

   a) $P(3,1)$ and $2x - y = 3$

   b) $P(1,1,1)$ and $< t + 1, 2 - 3t, 2t >$

   c) $P(1,1,1)$ and $x + y + z = 1$

   d) $< 2t - 2, 3 - 2t, t >$ and $2x + 3y + 2z = 4$

   e) $2x + 3y + 2z = 4$ and $4x + 6y + 4z = -10$

   f) $< t + 1, 2 - 3t, 2t >$ and $< 3 - 3t, 2 + 9t, 4 - 6t >$