Panel 1

Last Time

Vectors in Polar Coordinates $x = r\cos\theta$ $y = r\sin\theta$ π^2

Dot Product: $v = \langle v_1, v_2, v_3 \rangle$, $w = \langle w_1, w_2, w_3 \rangle$

$v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$

$\vec{v} \cdot \vec{v} = \|v\|^2$

$\vec{v} \cdot \vec{w} = \|v\| \|w\| \cos(\theta)$

Panel 2

Ex: Find angle between $u = i - 2j + 2k = \langle 1, -2, 2 \rangle$ and

a) $v = -3i + 6j + 2k = \langle -3, 6, 2 \rangle$

$u \cdot v = -3 - 12 + 4 = -11$

$\|u\| = \sqrt{9} = 3$

$\|v\| = \sqrt{49} = 7$

$\cos(\theta) = \dfrac{u \cdot v}{\|u\| \|v\|} = \dfrac{-11}{21}$

b) $w = 2i + 7j + 6\vec{k}$

$u \cdot w = \langle 1, -2, 2 \rangle \cdot \langle 2, 7, 6 \rangle = 2 - 14 + 12 = 0$

$\Leftrightarrow u \perp w$ (perpendicular)

Panel 3

Corollary: Two vectors $\vec{v}$ and $\vec{w}$ are perpendicular iff $\vec{v} \cdot \vec{w} = 0$

Ex: Which of the following vectors are perpendicular?

a) $\langle 1, 2, 3 \rangle$ and $\langle -1, -2, -3 \rangle$

b) $\langle 1, 2, 3 \rangle$ and $\langle -1, -3, 2 \rangle$

c) $\langle 1, 2, 3 \rangle$ and $\langle 6, -1, 1 \rangle$

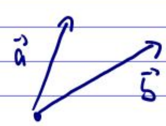
d) $\langle 1, 2, 3 \rangle$ and $\langle 5, -1, -1 \rangle$

e) $\langle 1, 2, 3 \rangle$ and $\langle 0, -3, 2 \rangle$

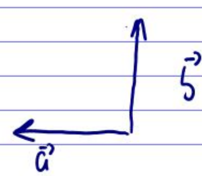
Panel 4

Picture Problems

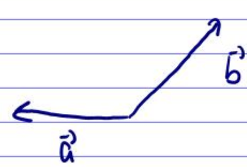
$$\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} = \cos\theta \iff \vec{a} \cdot \vec{b} =$$



$\vec{a} \cdot \vec{b}$: positive / zero / negative



$\vec{a} \cdot \vec{b}$: positive / zero / negative

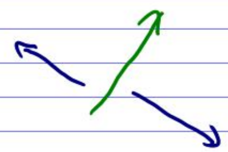


$\vec{a} \cdot \vec{b}$: positive / zero / negative

Panel 5

How many vectors are perp. to $\langle 2,3\rangle$ in $\mathbb{R}^2$?
Find them:

$$\langle 2,3\rangle\cdot\langle -3,2\rangle=0$$
$$\langle 3,-2\rangle=0$$



How many vectors are perp. to $\langle 1,2,3\rangle$ in $\mathbb{R}^3$.
Find a few of them

$$\langle 1,2,3\rangle\cdot\langle 0,-3,2\rangle$$
$$\langle 3,0,-1\rangle$$
$$\langle 2,-1,0\rangle$$

Panel 6

Ex: Find the angle that $\vec{a}=\langle 1,2,3\rangle$ makes with the y-axis:

y-axis is $\langle 0,1,0\rangle = \vec{j}$

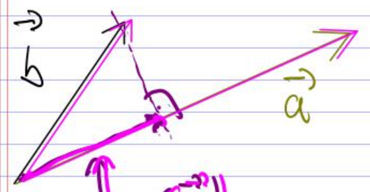
$$\frac{\langle 1,2,3\rangle\cdot\langle 0,1,0\rangle}{\sqrt{14}\quad 1} = \frac{2}{\sqrt{14}} = \cos(\theta)$$

Directional Angles of $\vec{v}=\langle v_1,v_2,v_3\rangle$ are:

$$\cos(\theta_x) = \frac{v_1}{\|v\|}$$
$$\cos(\theta_y) = \frac{v_2}{\|v\|}$$
$$\cos(\theta_z) = \frac{v_3}{\|v\|}$$

Panel 7

General Question: take two vectors $\vec{a}$ and $\vec{b}$. How much of $\vec{b}$ goes in the direction of $\vec{a}$?



$\operatorname{proj}_{\vec{a}}(\vec{b})$

$$x = \|\vec{b}\|\cos(\theta) = \cancel{\|\vec{b}\|}\,\frac{a\cdot b}{\|a\|\cancel{\|b\|}} = \frac{a\cdot b}{\|a\|}$$

Length of $\|\operatorname{proj}_{\vec{a}}(\vec{b})\| = \dfrac{a\cdot b}{\|a\|}$

Projection vector $\operatorname{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}}{\|a\|}\left(\dfrac{a\cdot b}{\|a\|}\right) = \dfrac{\vec{a}\cdot\vec{b}}{\|a\|^2}\,\vec{a}$

Panel 8

Projection Formula: $\operatorname{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\,\vec{a}$

Ex: Find length and direction of projection of $\vec{b} = \langle 1,1,2\rangle$ onto $\vec{a} = \langle -2,3,1\rangle$

$$\operatorname{proj}_{\vec{a}}(\vec{b}) = \frac{a\cdot b}{\|a\|^2}\,\vec{a} = \frac{\langle 1,1,2\rangle\cdot\langle -2,3,1\rangle}{(\sqrt{14})^2}\langle -2,3,1\rangle = \frac{3}{14}\cdot\langle -2,3,1\rangle$$

$$\operatorname{proj}_{\vec{b}}(\vec{a}) = \frac{a\cdot b}{\|b\|^2}\,b = \frac{3}{(\sqrt{6})^2}\langle 1,1,2\rangle = \frac{3}{6}\langle 1,1,2\rangle$$

Panel 9

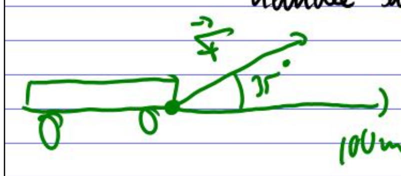
Projection Formula: $\text{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\,\vec{a}$

Ex: Find length and direction of projection of
$\vec{b} = \langle 1,1,2\rangle$ onto $\vec{a} = \langle -2,3,1\rangle$

$$\|\text{proj}_{\vec{a}}(\vec{b})\| = \frac{|a\cdot b|}{\|a\|}$$

Panel 10

Application: A wagon is pulled a distance of 100 m by a constant force of 70 N, applied to a handle held at 35°. Find work done by F.



$\|F\| = 70$ $\quad \vec{d}$ in dir. x

Recall: Work is $\vec{F}\cdot$ dist

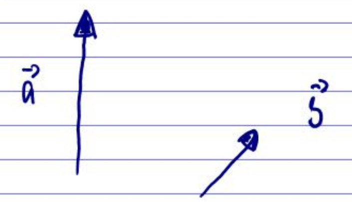
$$W = \|\text{proj}_{\vec{d}}(\vec{F})\|\ 100\text{m}$$

$$\vec{F} = \|F\|\langle\cos\theta, \sin\theta\rangle = \langle 70\cos(35), 70\sin(35)\rangle$$

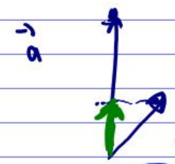
Work: $\dfrac{\langle 70\cdot\cos(35), 70\sin(35)\rangle\cdot\langle 1,0\rangle}{\|\langle 1,0\rangle\|}\cdot 100 = \underline{70\cos(35)\cdot 100}$

Panel 11

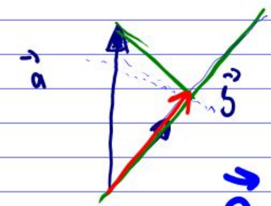
Picture Problems



find $\text{proj}_{\vec{a}}(\vec{b})$ $\text{proj}_{\vec{b}}(\vec{a})$



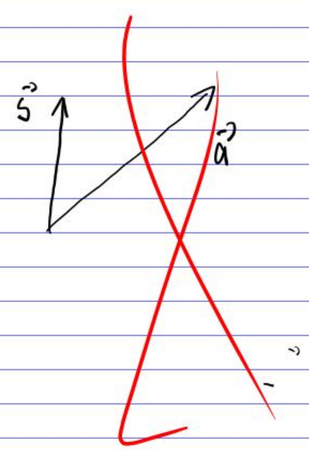
$$\frac{\vec{a} \cdot \vec{b}}{||a||^2} \vec{a}$$



$$\frac{\vec{a} \cdot \vec{b}}{||b||^2} \vec{b}$$

Panel 12

It looks like



Panel 13

So: Add / Subtract vector → ✓

Dot product of vectors → + angles

− no vector

Cross Product: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

then

$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$

+ : gives vector

− : complicated,

only in $\mathbb{R}^3$ or $\mathbb{R}^2$

$(a_1, a_2, 0)$

Panel 14

How to memorize the cross product:

$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$

$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1 \rangle$

Ex: $\langle 1, 3, 4 \rangle \times \langle 2, 7, -5 \rangle =$

$\begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \langle -15-28, -(-5-8), 7-6 \rangle$

$\langle -43, 13, 1 \rangle$

Panel 15

$\langle 1,0,-2\rangle \times \langle 0,2,-3\rangle$

$$\begin{matrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 2 & -3 \end{matrix} \qquad \langle 0-{-4}, -(-3-0), 2-0\rangle$$

$$\langle 4,3,2\rangle \in \mathbb{R}^3$$

$\langle 1,0,0\rangle \times \langle 0,1,0\rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0,0,1\rangle \qquad \vec{i}\times\vec{j} = \vec{k}$$

$\langle 1,0,-2\rangle \cdot \langle 0,2,-3\rangle = 6 \in \mathbb{R}$

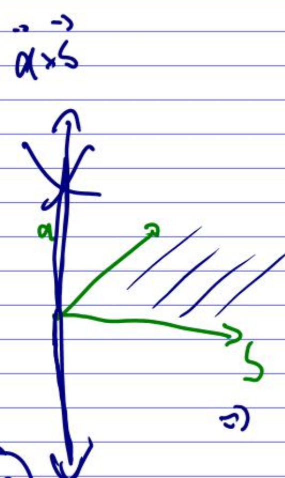
Panel 16

Properties: (1) $\vec{a}\times\vec{a} = \vec{0}$

(2) $\vec{a}\times\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$, using right-hand rule:

bend fingers of right hand from a to b

$\Rightarrow \vec{a}\times\vec{b}$ is your thumb



$\Rightarrow \vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$ anti-commutative

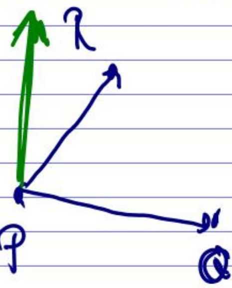
7

Panel 17

Ex: Find vector perpendicular to the plane through $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$

3 points form a plane

2 vectors form a plane:



$\vec{PQ} = (-2,5,-1) - (1,4,6) = \langle -3,1,-7 \rangle$

$\vec{PR} = \langle 0,-5,-5 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} =$$

HW