Panel 1

Contour Integration

Contour Integrals

function f

does not quite fit.

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x')^2+(y')^2}\,dt$$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

vector field $\vec{F}$

Goal

$$\int_C \vec{F}\,d\vec{r}$$

def: $\int_C M\,dx + N\,dy$

conservative: $f(B) - f(A)$

2D, closed curve C: $\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$

Panel 2

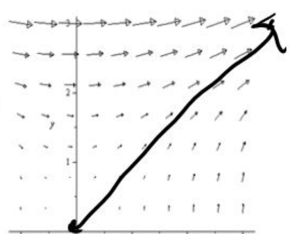
Find the following integrals. Explain your reasoning.

a) $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = <2xy^3 + y\sin(x), 3x^2y^2 - \cos(x)>$ and C is the boundary of the square with corner point (0,0), (1,0), (1,1), and (0, 1), oriented counter-clockwise.

b) $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = <2xy^2, 2x^2y>$ and C a curve from (0,0) to (1,1). $= f(1,1) - f(0,0) = 1$

c) $\int_C (\cos(x^2) - 4y)dx + (e^y - 3x)dy$, where C is the unit circle.

d) $\int_C \vec{F}\cdot d\vec{r}$ (sign only, i.e. positive, negative, or zero) where $\vec{F}$ is the vector field shown and C is the line segment from (0,0) to (3,3)



a) $\iint [6xy^2 + \sin(x)] - [6xy^2 + \sin(x)]\,dA = 0$

b) $f(x,y) = x^2y^2$

c) $\iint_R -3 - (-4)\,dA = \iint_R dA = \text{area(unit circle)} = \pi$

Panel 3

Alternate version of Green's Theorem: C closed in $\mathbb{R}^2$

$$\oint_C \vec{F}\, d\vec{r} = \oint_c M\,dx + N\,dy = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

Consider $F = \langle M, N \rangle = \langle M, N, 0 \rangle$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix}$$

$$\Rightarrow \text{curl}(F) = \left\langle 0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

Green's theorem:

$$\oint_C \vec{F}\, d\vec{r} = \iint_D \text{curl}(\vec{F}) \cdot \vec{k}\, dA$$

$$\vec{k} = \langle 0, 0, 1 \rangle$$

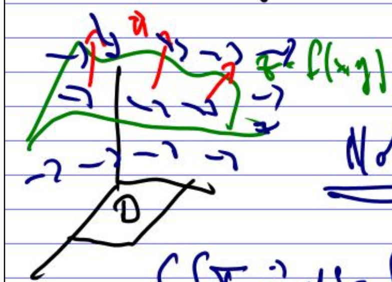
Panel 4

Def: If S is surface given by $z = f(x,y)$, over region R.

$\Rightarrow \iint_S \vec{F} \cdot \vec{n}\, dS$ is the flux of $\vec{F}$ through S

where $\vec{n}$ is the normal vector to the surface S

given by $\vec{n} = \langle -f_x, -f_y, 1 \rangle \dfrac{1}{\sqrt{f_x^2 + f_y^2 + 1}}$



Note: $dS = \sqrt{f_x^2 + f_y^2 + 1}\, dA$

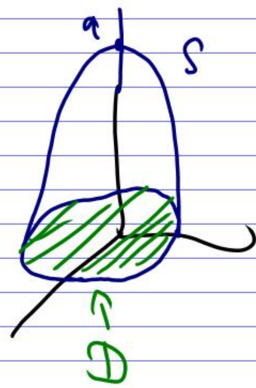
$$\iint_S F \cdot \vec{n}\, dS = \iint_S \langle M, N, P \rangle \cdot \vec{n}\, dS =$$

$$\iint_D -f_x M - f_y N + P\, dA \quad (z = f(x,y))$$

Panel 5

Ex: Let S be $z = 9 - x^2 - y^2$, $z \geq 0$ and $\vec{F} = \langle 3x, 3y, z \rangle$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad M \quad N \quad P$

Find flux of $\vec{F}$ through S.



$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D -f_x M - f_y N + P\, dA$$

$z = f(x,y) = 9 - x^2 - y^2$

$$= \iint_D 2x \cdot 3x + 2y\, 3y + (9 - x^2 - y^2)\, dA$$

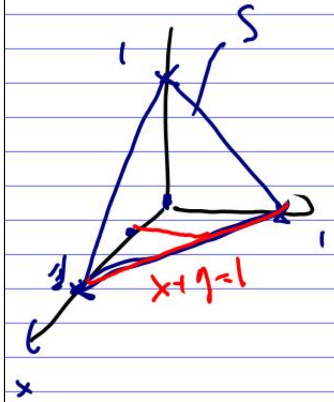
$$= \iint_D 9 + 5(x^2 + y^2)\, dA = \int_0^{2\pi}\int_0^{3} (9 + 5r^2)\, r\, dr\, d\theta$$

Panel 6

Ex: Let S be the surface defined by $x + y + z = 1$ solid by the coordinate planes and let $\vec{F}$ be the vector field $\vec{F} = \langle x, xy, xyz \rangle$. Find the flux of $\vec{F}$ through S.
$\qquad\qquad\qquad\qquad M \quad N \quad P$

$z = 1 - x - y$

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D -f_x M - f_y N + P\, dA$$



$$= \iint_D x + xy + xy(1 - x - y)\, dA$$

$$= \int_0^1 \int_0^{1-x} x + xy + xy(1 - x - y)\, dy\, dx = \checkmark$$

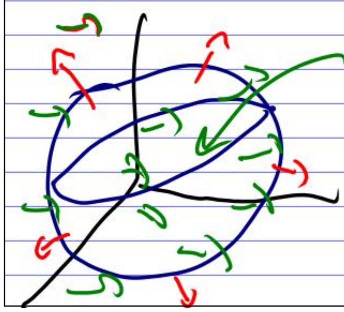
Panel 7

The Divergence Theorem:

3D

Let Q be a region in $\mathbb{R}^3$ bounded by a closed surface S with outward normal $\vec{n}$. If $\vec{F}$ is a 3D vector field with cont. derivatives, then

$$\oiint_S \vec{F}\cdot\vec{n}\, dS = \iiint_Q \operatorname{div}(\vec{F})\, dV$$

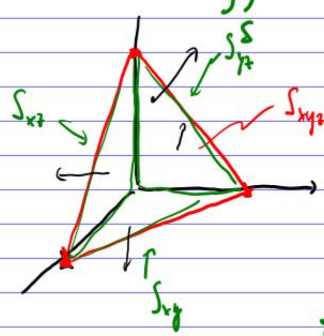


Q is inside of closed surface S.

Panel 8

Ex: Let S be the closed surface bounded by $x+y+z=1$ and by the coordinate planes. Let $\vec{F}$ be the vector field $\vec{F} = \langle x, xy, xyz \rangle$, as before. Find flux of $\vec{F}$ through the closed surface S:

$$\iint \vec{F}\cdot\vec{n}\, dS = \iint_{S_{xyz}} \vec{F}\cdot\vec{n}\, dS + \iint_{S_{yz}} + \iint_{S_{xz}} + \iint_{S_{xy}}$$

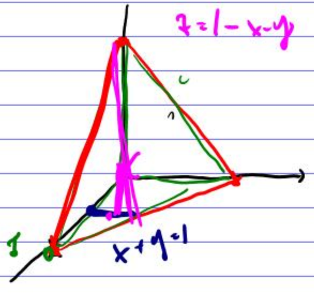


use Div. thm
to simplify to ONE int'l

Panel 9

$\vec{F} = \langle x, xy, xyz \rangle$,

$$\oiint \vec{F}\cdot\vec{n}\, dS = \iiint_Q div(F)\, dV$$



$$= \iiint_Q 1 + x + xy\, dV$$

could include z's

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+xy)\; dz\, dy\, dx$$

(try this first!) = Maple or W.A.

Panel 10

Compare Green's Thm and Divergence Thm:

Green: $\vec{F}$ 2D vector field, C closed curve

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

$$= \iint_D curl(\vec{F})\cdot\vec{k}\, dA$$

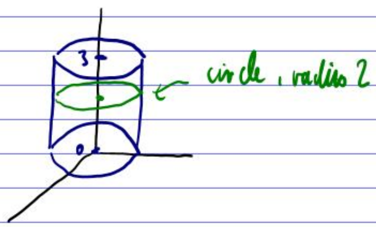
Divergence: $\vec{F}$ 3D vector field, S closed surface

(Gauss Thm) $$\oiint_S \vec{F}\cdot\vec{n}\, dS = \iiint_Q div(\vec{F})\, dV$$

Panel 11

Ex: Let S be the surface defined by $x^2+y^2=4$, $z=0$, and $z=3$. If $\vec{F}=\langle x^3, y^3, z\rangle$ find flux of $\vec{F}$ through S.

$\iint_S \vec{F}\cdot\vec{n}\,dS$ = 3 integrals or ??



HW