Panel 1

Last Time

Function $f(x,y)$ —
- $\int f(x,y)\, ds$
- $\int f(x,y)\, dx$
- $\int f(x,y)\, dy$

Vector field $\vec{F} = \langle M, N \rangle$ — $\int_C \vec{F}\, d\vec{r} = \int_C M\,dx + N\,dy$

Fundamental Thm. of Line Integration:

Panel 2

Fundamental Theorem for Line Integrals

If $\vec{F}$ is conservative with potential function f, and $\vec{r}(t)$, $a \le t \le b$, a smooth curve. Then:

$$\int_\gamma \vec{F}\, d\vec{r} = f(\gamma(b)) - f(\gamma(a)) = f(B) - f(A)$$

Consequences: If $\vec{F}$ is conservative then:

① $\int_C \vec{F}\, d\vec{r}$ independent of path from A to B

② $\oint_C \vec{F}\, d\vec{r} = 0$, C closed curve

Panel 3

Ex: Let $F(x,y) = \langle \frac{y^2}{1+x^2}, 2y \arctan(x) \rangle$ and

$r(t) = \langle t^2, 2t \rangle$, $t \in [0,1]$. Find $\int_r \vec{F}\,d\vec{r}$

① $\int_C \vec{F}\,d\vec{r} = \int \frac{y^2}{1+x^2} dx + 2y \arctan(x)\, dy =$

$\int \frac{(2t)^2}{1+t^4} 2t\,dt + 2 \cdot 2t \cdot \arctan(t^2)\, 2dt = ?!!$

② $N_x = \frac{2y}{1+x^2} = M_y \Rightarrow \vec{F}$ is conservative

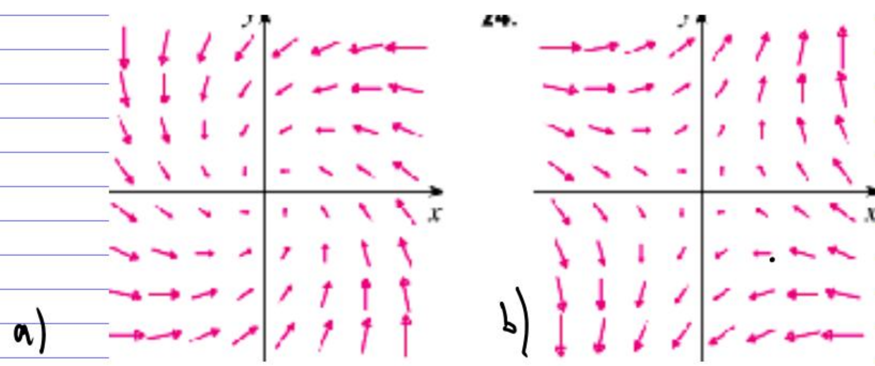
$f_x = \frac{y^2}{1+x^2} \Rightarrow f = y^2 \arctan(x)$ ✓ $\int \vec{F}\,d\vec{r} = y^2 \arctan(x) \Big|_{(0,0)}^{(1,2)} = 4 \arctan(1) = 4\frac{\pi}{4} = \pi$

$f_y = 2y \arctan(x)$ ✓

Panel 4

Quiz #10 Name: ______

① Which vector field is conservative?



② Is $\int_C y^2 - 1\,dx + 2xy\,dy$ independent of the path C from A to B? YES NO

Panel 5

(3) Find the work done moving a particle along a straight line from (0,1) to (3,2) through the vector field $\vec{F}(x,y) = \langle 3x^2 + y^2, 2xy \rangle$

(4) Find $\oint_C \tan(y)\,dx + x\sec^2(y)\,dy$ where $r(t) = \langle \cos(t), \sin(t) \rangle$, $t \in [0, 2\pi]$

Panel 6

<u>Work Integral</u> $\int \vec{F}\,d\vec{r}$

$\int_C \vec{F}\,d\vec{r}$ → do it. $\int M\,dx + N\,dy$
↘ potential function if conservative

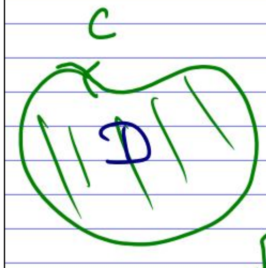
$\oint_C \vec{F}\,d\vec{r}$ < do it
< 0 ($\vec{F}$ is conservative)

short-cut →

Panel 7

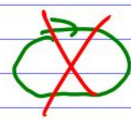
Green's Theorem: R a region in xy-plane with boundary curve C. C is piecewise smooth, non-intersecting, closed, and positively oriented. $\vec{F} = (M, N)$ is a smooth vector field. Then:

$$\oint_C \vec{F}\, d\vec{r} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$



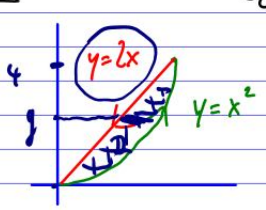
Corollary: If $\vec{F}$ is conservative $\Rightarrow \oint_C \vec{F}\, d\vec{r} = 0$

pos orientation: inside is on left



Panel 8

Ex: Evaluate $\oint_C 5xy\,dx + x^3\,dy$, where C is as shown:



Method A: Green's theorem

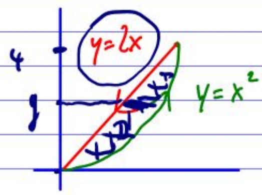
$$\oint_C 5xy\,dx + x^3\,dy = \iint_D N_x - M_y \, dA =$$

$$= \iint_D 3x^2 - 5x \, dx\,dy =$$

$$= \int_0^4 \int_{y/2}^{\sqrt{y}} 3x^2 - 5x \, dx\,dy = \int_0^4 x^3 - \frac{5}{2}x^2 \Big|_{y/2}^{\sqrt{y}} dy = -\frac{28}{15}$$

B: $\int_{y=x^2} + \int_{y=2x} =$

Panel 9



$r(t) = \langle t, t^2 \rangle, \ t \in [0,2]$

$r(t) = \langle t, 2t \rangle, \ t \in [2,0]$

$$\int_C 5xy\,dx + x^2\,dy = \int_0^2 5\cdot t\cdot t^2\,dt + t^3\,2t\,dt =$$

$$\int_2^0 5t\,2t\,dt + t^2\,2dt =$$

$-\frac{21}{15}$

Panel 10

Ex: Evaluate $\oint_C 2xy\,dx + (x^2+y^2)\,dy$, C is $4x^2+9y^2=36$

Green:

$$\oint_C \_\_ = \iint_D N_x - M_y\,dA = \iint_D 2x - 2x\,dA$$

2

$= 0$

Panel 11

Ex: Find $\oint_\gamma (x\sin(y^2) - y)\,dx + (x^2 y\cos(y^2) + 3x)$ (N_x)

where γ is the triangle $(0,0)$, $(1,0)$, $(0,1)$.

By Green: $\oint_\gamma = \iint_D 2xy\cos(y^2) + 3 - (2xy\cos(y^2) - 1)\, dA$

$$= \iint_\Delta 2\,dA = 2\iint_\Delta dA = 2\,\text{area}(\Delta) = 2\cdot\frac{1}{2} = 1$$

Panel 12

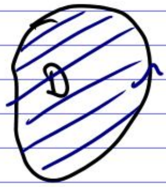
Evaluate $\oint_C \left(3y - e^{\sin(x)}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$ (*) (N_x)

where C is the circle $x^2 + y^2 = 9$

(*) $= \iint 7 - 3\, dA = \iint_{\text{circle}} 4\, dA = \pi(3)^2 \cdot 4 =$

Panel 13

Theorem: If D is a region enclosed by a curve C then $\text{area}(D) = \frac{1}{2}\oint_C x\,dy - y\,dx$



Proof.

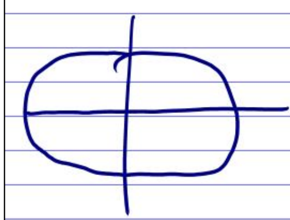
$$\oint_C \overset{N}{x}\,dy - \overset{M}{y}\,dx = \iint 1-(-1)\,dA$$

$$= \iint 2\,dA =$$

$$= 2\cdot\text{area}$$

Panel 14

Ex: Find area enclosed by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$A = \frac{1}{2}\oint_C x\,dy - y\,dx =$$

$$= \frac{1}{2}\int_0^{2\pi} a\cos(t)\, b\cos(t) - b\sin(t)(-a\sin(t))\,dt$$

$r(t) = \langle a\cos(t), b\sin(t)\rangle$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2(t)+\sin^2(t))\,dt =$$

$$= \frac{1}{2}2\pi ab = \pi\cdot ab$$