Panel 1

$$\int_C x^2 y\,dx = \int_0^1 (2t)^2 (3t-1)\,2\,dt \qquad C: r(t) = \langle \overset{x}{2t}, \overset{y}{3t-1} \rangle, \; t \in [0,1]$$

$x = 2t$, $\quad dx = 2\,dt$,

$y = 3t-1$, $\quad dy = 3\,dt$

$$\int_C x^2\,dx + y\,dy = \int_0^1 (2t)^2\,2\,dt + (3t-1)\,3\,dt =$$

$$= \int_0^1 (2t)^2 \cdot 2 + (3t-1)3\,dt = \underline{\underline{\frac{25}{6}}}$$

Panel 2

<u>Fundamental Theorem of Line Integration</u>

Suppose $\vec{F}$ is a conservative vector field. Then

$$\int_C \vec{F}\,d\vec{r} = \int_C M\,dx + N\,dy$$

$$= f(b) - f(a), \quad C \text{ goes from } a \text{ to } b$$

f is potential funct.

<u>Proof</u>: $\int \vec{F}\,d\vec{r} = \int M\,dx + N\,dy = \qquad \Rightarrow f_x = M, \; f_y = N$

$$= \int f_x\,dx + f_y\,dy = \int \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}\,dt + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\,dt =$$

$$= \int \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right)dt = \int_a^b \frac{\partial f}{\partial t}\,dt = f(b) - f(a)$$

Panel 3

Ex: Find work done by gravitational field $\vec{F}(\vec{r}) = -\frac{mMG}{\|r\|^3}\vec{r}$ moving particle from $(3,4,12)$ to $(2,2,0)$.
$A \qquad\qquad B$

$$\boxed{W = \int_A^B F\,dr = f(2,2,0) - f(3,4,12)}$$

Method 1: parametric equation of curve from A to B

BUT path is not specified, so this is no good

$\Rightarrow \vec{F}$ got to be conservative!!!

$f(x,y,z) = mMG\,(x^2+y^2+z^2)^{-1/2}$

$f_x = -\frac{1}{2} mMG\,(x^2+y^2+z^2)^{-3/2}\,2x$

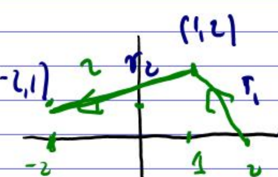
$f_y = -\frac{y\,mMG}{(x^2+y^2+z^2)^{3/2}}$

$f_z = \ldots$

$$\langle f_x, f_y, f_z\rangle = -mMG\,(x^2+y^2+z^2)^{-3/2}\langle x,y,z\rangle = -\frac{mMG}{\|r\|^3}\vec{r}$$

Panel 4

Evaluate $\int_C y^2\,dx + 2xy\,dy$.



Method 1: $r_1(t) = (2,0) + t\langle -1, 2\rangle = \langle 2-t, 2t\rangle$, $t \in [0,1]$

$$\Rightarrow \int_{r_1} (2t)^2(-1) + 2(2-t)(2t)\,2\,dt = 4$$

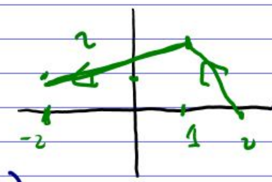
$r_2(t) = \langle 1,2\rangle + t\langle -3,-1\rangle = \langle 1-3t, 2-t\rangle$, $t\in[0,1]$

$$\Rightarrow \int (2-t)^2(-3) + 2(1-3t)(2-t)(-1)\,dt = -6$$

$$W = \int_C y^2 dx + 2xy\,dy = 4 - 6 = \underline{\underline{-2}}$$

Panel 5

Evaluate $\int_C y^2\,dx + 2xy\,dy$.



$W = \int \vec{F}\,d\vec{r}$, where $\vec{F} = \langle y^2, 2xy \rangle$

$$\frac{\partial N}{\partial x} = 2y = \frac{\partial M}{\partial y} \checkmark$$

$f(x,y) = xy^2 + c$ Thus, by Fund. Theorem

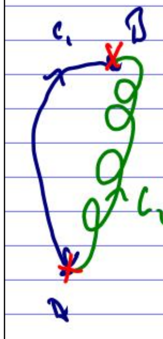
$$\int_C y^2\,dx + 2xy\,dy = f(B) - f(A) = xy^2 \Big|_{(2,0)}^{(-2,1)} = -2 + 0 = \underline{\underline{-2}}$$

Panel 6

Thm: If $\vec{F} = \nabla f$, i.e. $\vec{F}$ is conservative and C is a smooth curve from A to B then

$$\int_C \vec{F}\,d\vec{r} = f(B) - f(A)$$

Corollary: C_1 a curve from A to B and C_2 another curve from A to B, and $\vec{F}$ is conservative, then



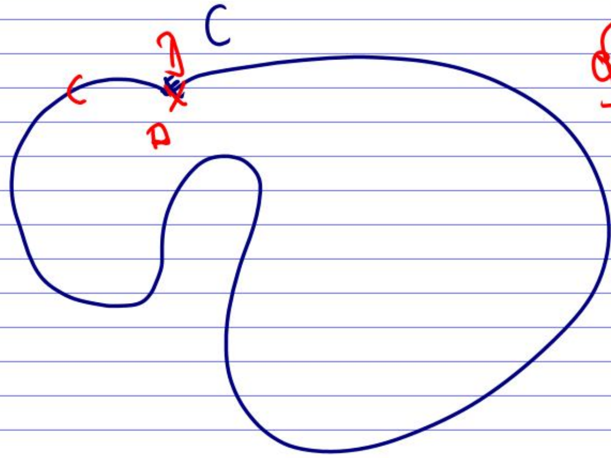
$$\int_{C_1} \vec{F}\,d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r}$$

Panel 7

Corollary 2: If $\vec{F}$ is conservative and C a closed curve then
$$\oint_C \vec{F}\,d\vec{r} = 0$$

Note: if C is closed we write $\oint_C \vec{F}\,d\vec{r}$

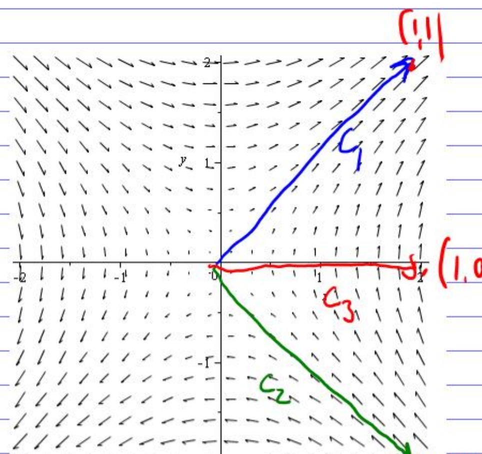
Proof



$$\oint \vec{F}\,d\vec{r} = f(A) - f(A) = 0$$

Panel 8

Ex: Are the following integrals positive or negative?



$$\int_{C_1} \vec{F}\,d\vec{r} = (xy)\Big|_{(0,0)}^{(1,1)} = 1$$

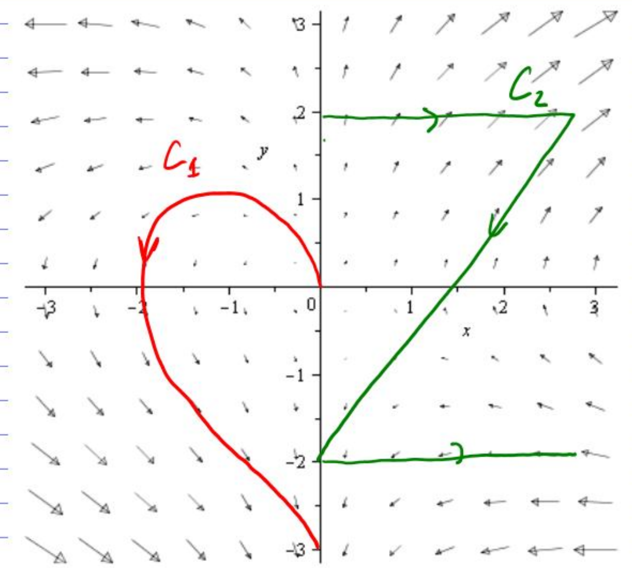
$$\int_{C_3} \vec{F}\,d\vec{r} = (xy)\Big|_{(0,0)}^{(1,0)} = 0$$

$$\int_{C_2} \vec{F}\,d\vec{r} = (xy)\Big|_{(0,0)}^{(1,-1)} = -1$$

$\vec{F} = \langle y, x \rangle \Rightarrow f(x,y) = xy$
potential

Panel 9

Ex: Are the following integrals positive or negative?



$\int_{C_1} \vec{F}\, d\vec{r}$ pos

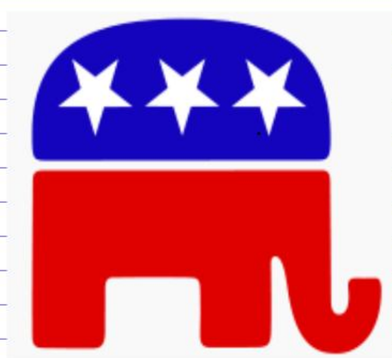
pos or neg

$\int_{C_2} \vec{F}\, d\vec{r}$ neg

pos or neg

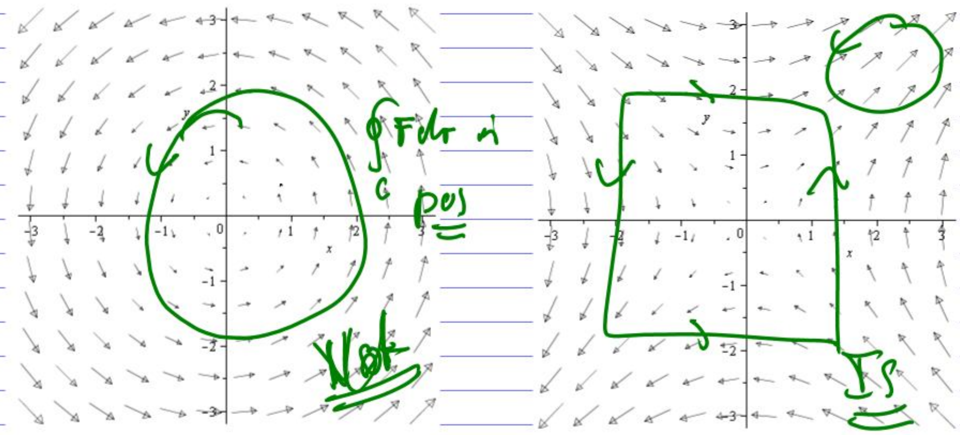
Panel 10

Which of the following vector fields look conservative?



Panel 11

Which of the following vector fields looks conservative?



If $\vec F$ conservative, then

$$\oint_C \vec F \, d\vec r = 0$$

Panel 12

Find a conservative vector field that has the given potential:

$f(z, y, z) = \sin(x^2 + y^2 + z^2)$

$\boxed{\cos(x^2+y^2+z^2)\langle 2x, 2y, 2z\rangle}$

Find $div(\nabla \cdot F)$ and $curl(F) = \nabla \times F$

$F(x, y, z) = < x^2z, y^2x, y + 2z >$

$div(\vec F) = 2xz + 2xy + 2$

Evaluate $\int_C (x - y)dx + xdy$ if C is the graph of $y^2 = x$ from (4,-2) to (4,2)

old-Construction $r(t) = \langle t^2, t \rangle$, —

Find the work done by F(x,y,z) along the curve $< t, t^2, t^3 >$ from (0, 0, 0) to (2, 4, 8), where $F(x, y, z) = < y, z, x >$

$\int \langle y, z, x\rangle \cdot d\vec r = \int y\,dx + z\,dy + x\,dz = \int_0^2 t^2\,dt + t^3\,2t\,dt + t\,3t^2\,dt$

Check which of the following vector fields is not conservative.

$F(x, y) = < 3x^2y + 2, x^3 + 4y^3 >$

$F(x, y) = < e^x, 3 - e^x \sin(y) >$

$F(x, y, z) = < 8xz, 1 - 6yz^2, 4x^2 - 9y^2z^z >$

Show that the line integrals are independent of the path, and find their value:

$\int_{(-1,2)}^{(3,11)} (y^2 + 2xy)dx + (x^2 + 2xy)dx$

$f = xy^2 + x^2y$

$\int_{(1,0,2)}^{(-2,1,3)} (6xy^3 + 2z^2)dx + (9x^2y^2)dy + (4xz + 1)dz$

Panel 13

$\int_C \vec{F}\,d\vec{r}$ important : Work

$\int_C \vec{F}\,d\vec{r}$ — old fashioned : $\int M\,dx + N\,dy$, $r(t) = (x(t), y(t))$
— if $\vec{F}$ is cons. $f(B) - f(A)$

$\oint_C \vec{F}\,d\vec{r}$ — old fashion
— if $\vec{F}$ conservative
— 0

Next time!!!