Panel 1

Last Time:

$+z_0 = f(x,y)$

Tangent Plane to $f(x,y)=z$: $f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

PDE: $\Delta u = u_{xx}+u_{yy}=0$

Directional deriv. : $D_{\vec{u}}(f)=(\nabla f)\cdot\vec{u}$, $\|\vec{u}\|=1$

Gradient and its Properties

$\nabla f = \langle f_x, f_y\rangle$

$\nabla^2 f = f_{xx}+f_{yy}$

Panel 2

Properties of Gradient

- The gradient is a vector
- Gradient is perpendicular to level curves
- Gradient points in direction of largest increase
- $\|\nabla f\|$ is the largest increase

Ex: Find ∇f if $f(x,y,z)=\ln(xy^2z^3)$

$$\nabla f = (f_x, f_y, f_z) = \left\langle \frac{1}{xy^2z^3}\cdot y^2z^3, \frac{1}{xy^2z^2}\cdot xz^3, \frac{1}{xy^2z^3}\cdot xy^2\right\rangle$$

$$= \left\langle \frac{1}{x}, \frac{1}{y^2}, \frac{1}{z^3}\right\rangle$$

Find max. rate of change for $f(x,y,z)=\ln(xy^2z^3)$ at $P(1,1,1)$

$\|\nabla f\| = \| = \|(1,1,1)\| = \underline{\underline{\sqrt{3}}}$

Panel 3

Ex: Suppose the level curves of an area are given by $f(x,y) = y\ln(x)$. You are standing at $P(1,-3)$ and you are heading in the direction $\langle -4,3\rangle$.

① Are you going up or down? How much? ② Which way should you go for max change in height?

① $D_{\frac{\langle -4,3\rangle}{5}}(y\ln(x))\Big|_{(1,-3)} = \nabla f\Big|_{(1,-3)} \cdot \left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle = \left\langle \frac{y}{x}, \ln(x)\right\rangle\Big|_{(1,-3)} \cdot \left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle$

$\|\langle -4,3\rangle\| = 5$

$= \langle -3,0\rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle = \frac{12}{5}$

② $\nabla f\Big|_{(1,-3)} = \langle -3,0\rangle$, max rate of change is 3

Panel 4

Name: ____________

Quiz

① Consider $f(x,y) = x^2 + 3xy - y^2$. Find

a) $f_x = 2x + 3y$

$f_y = 3x - 2y$

b) $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x+3y) = 3$

$f_{yx} = 3$

c) $\nabla f = \langle 2x+3y, 3-2y\rangle$

② If $f(x,y) = xy + 3xy^2$. Find tangent plane at $P(1,1,4)$

$f_x = y + 3y^2\Big|_{(1,1)} = 4$

$f_y = x + 6xy\Big|_{(1,1)} = 7$

$p(x,y) = 4(x-1) + 7(y-1) + 4$

plot3d({___, ___}, x=-2..4, y=-2..4)

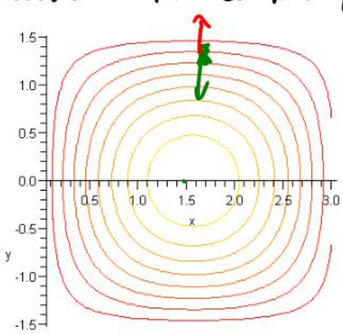
Panel 5

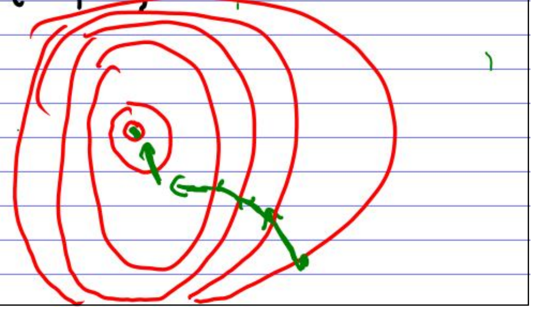
(3) $f(x,y) = x^3 - 3xy + 4y^2$. Find directional derivative in the direction of $\langle 3/5, 4/5 \rangle = \vec{u}$, $\|\vec{u}\| = 1$

$\nabla f = \langle 3x^2 - 3y, -3x + 8y \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{3}{5}(3x^2 - 3y) + \frac{4}{5}(-3x + 8y)$

(4) Consider the contour plot below. Sketch the gradient at

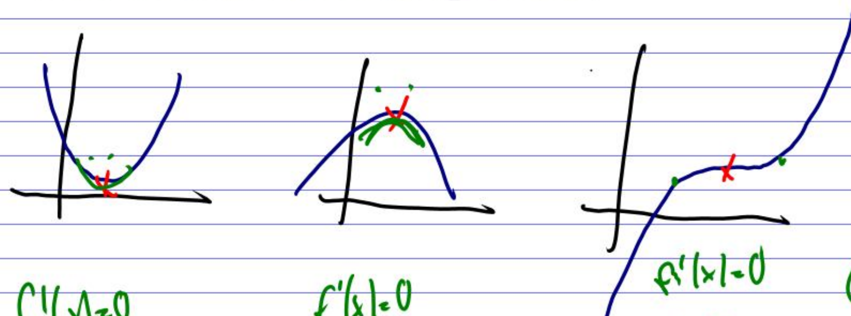
$P(1.5, 1.0)$





Panel 6

Review of Max/Min problems in $\mathbb{R}$



$f'(x) = 0$ $f'(x) = 0$ $f'(x) = 0$ (critical)

$f'' > 0$ $f'' < 0$ $f''(x) = 0$

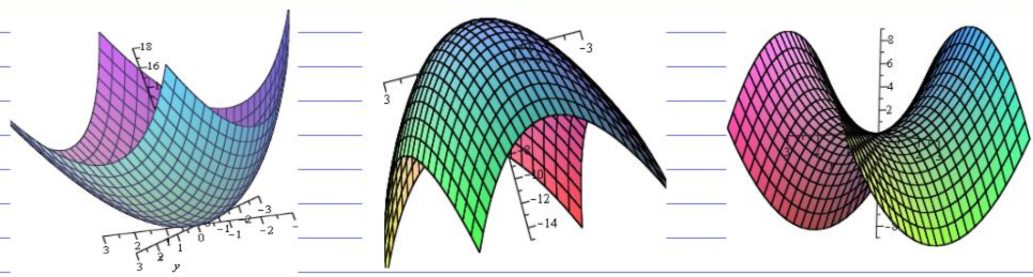
(1) f'

(2) critical points

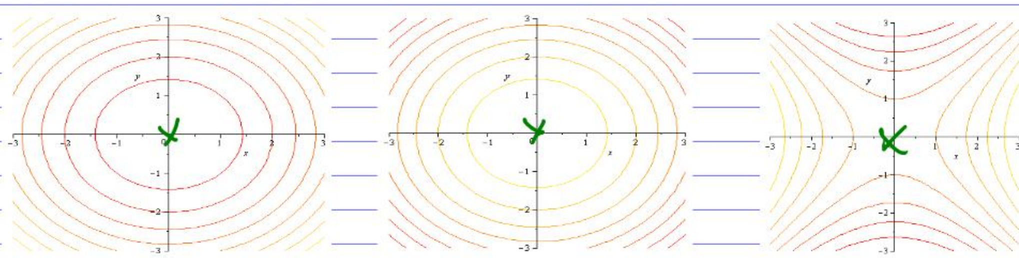
(3) Use test such as 2nd deriv. test

Panel 7

min | max | saddle



$f(x,y) = x^2+y^2$ | $f(x,y) = 1-x^2-y^2$ | $f(x,y) = x^2-y^2$



Panel 8

Max / Min Problems

To find max/min of $z = f(x,y)$:

① Find ∇f

② Solve $\nabla f = 0$ $(f_x = 0,\ f_y = 0)$

③ Compute $H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ and $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

a) f has min if: $D > 0$, $f_{xx} > 0$ → min

b) f has max if. $D > 0$, $f_{xx} < 0$, max

c) f has saddle if: $D < 0$

d) no information if: $D = 0$ no info

Panel 9

<u>Ex</u>: Find and classify the critical points for

$$f(x,y) = x^2 - 2xy + 3y^2 + 4x$$

(1) $f_x = 2x - 2y + 4 = 0$ (2) $4y + 4 = 0$, $y = -1$, $x = -3$ is a min

$f_y = -2x + 6y = 0$

$-2x - 6 = 0$

(3) $f_{xx} = 2$

$f_{yy} = 6$

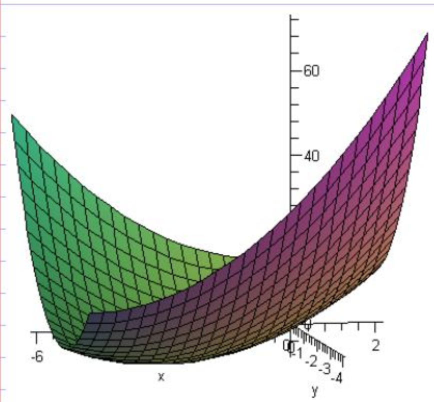
$f_{xy} = f_{yx} = -2$

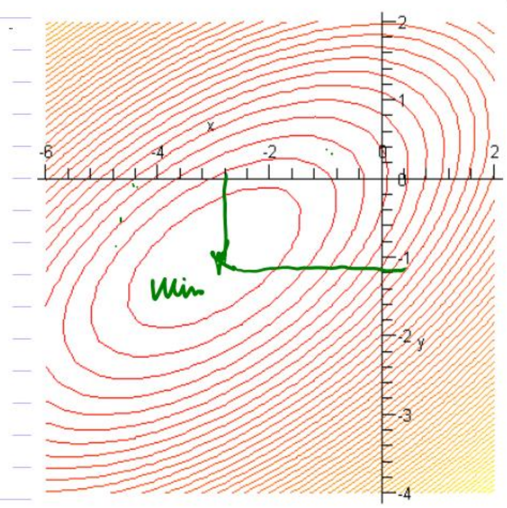
$$H = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}, \quad D = 2 \cdot 6 - (-2)^2 = 8 > 0$$

$f_{xx} > 0$

Panel 10

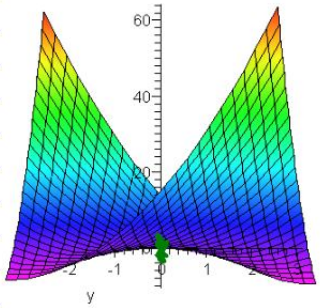
$$f(x,y) = x^2 - 2xy + 3y^2 + 4x$$





Panel 11

Suppose $f(x, y) = x^2 + 2y^2 + 4xy$. Find and classify all relative extrema, if any.



① $f_x = 2x + 4y = 0$

$f_y = 4y + 4x = 0$ 1-1

② $-2x = 0, \ x = 0, \ y = 0$

③ $H = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \Rightarrow D = 4 - 16 = -12 < 0$

(0,0) is a saddle point

Panel 12

Find and classify critical points for $f(x,y) = 3x - x^3 - 2y^2$

① $f_x = 3 - 3x^2 = 0$

$f_y = -4y = 0$

② $x = 1$ or -1

$y = 0 \ , \ 0$

(1,0) max

(-1,0) saddle

critical pts

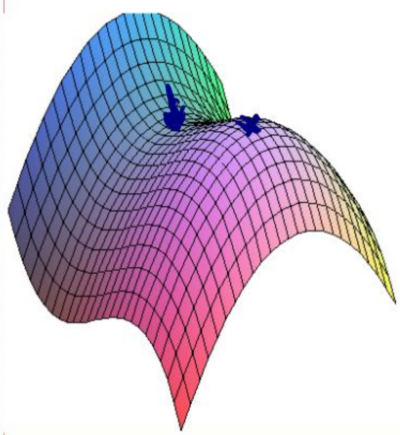
③ $H = \begin{pmatrix} -6x & 0 \\ 0 & -4 \end{pmatrix}, \ D = 24x$

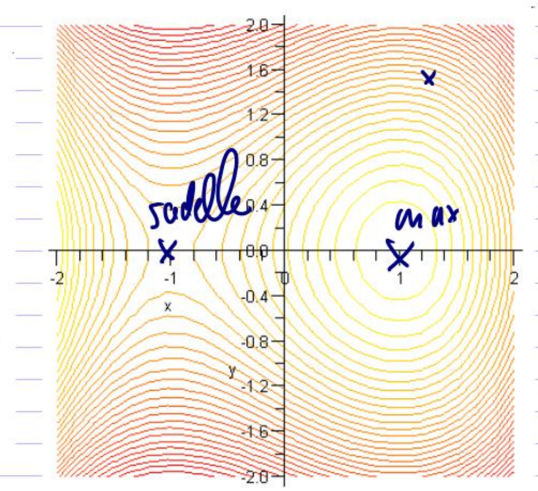
at $(1,0)$: $D > 0$, $f_{xx} = -6 \Rightarrow$ max

at $(-1,0)$: $D < 0$: saddle

Panel 13

$f(x,y) = 3x - x^3 - 2y^2$





Panel 14

Ex: Find and classify the critical points for $f(x,y) = x^3y + 12x^2 - 8y$

① $f_x = 3x^2y + 24x = 0$ $\quad 12y + 48 = 0 \quad , y = -4$

$f_y = x^3 - 8 = 0$ $\quad x = 2$

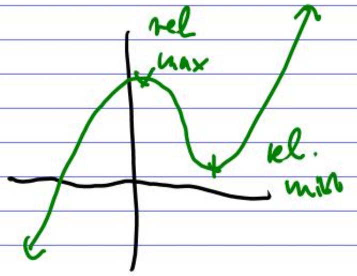
② $H = \begin{pmatrix} 6xy + 24 & 3x^2 \\ 3x^2 & 0 \end{pmatrix}$ $\quad D = -9x^4 \Big|_{(2,-1)} < 0$

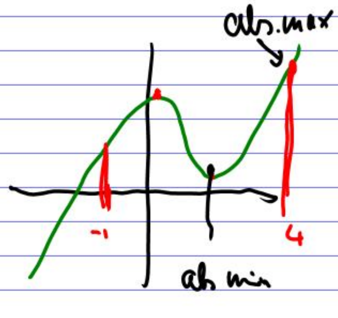
$\rightarrow$ saddle

Panel 15

Absolute Max / Min:

Differences between absolute and relative extrema





Thm f cont. on [a, b], then f has abs. max + abs. min!
They occur either at a critical point, or on the boundary of [a, b]

① $f' = 0$

②

x	f(x)
critical	?
a	?
b	?