Panel 1

The Chain Rule

$f: \mathbb{R} \to \mathbb{R},\ g: \mathbb{R} \to \mathbb{R} \Rightarrow \frac{d}{dx} f(\underbrace{g(x)}) = f'(g(x))\, g'(x)$

Chain Rule in $\mathbb{R}^2$:

$z = f(x,y),\ x = g(t),\ y = h(t)$

$$\Rightarrow \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$z = f(x,y),\ x = \underline{g(s,t)},\ y = h(s,t)$

$$\Rightarrow \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

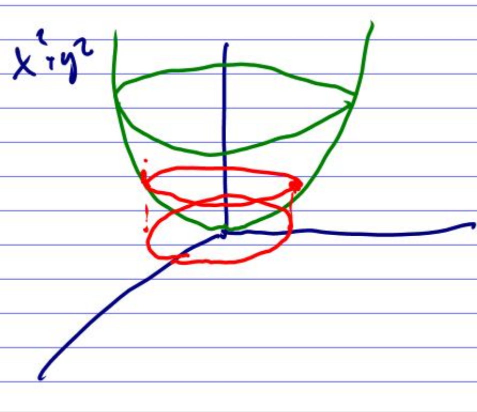
Panel 2

What does this mean?

$f(x(t), y(t)) = (\cos(t))^2 + (\sin(t))^2 = 1$

$f(x,y) = x^2 + y^2$ $\quad x = \cos(t),\ y = \sin(t)$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 2x\cdot(-\sin(t)) + 2y\cos(t) = -2\cos(t)\sin(t) + 2\sin(t)\cos(t) = 0$$



$r(t) = \langle \cos(t), \sin(t) \rangle$

circle radius 1 in xy-plane

$f(x(t), y(t))$ is cut-out along surface + according to curve $r(t)$

Panel 3

Ex: $f(x,y) = x^2 y + 3xy^4$, $x = \sin(2t)$ $y = \cos(t)$. Find $\frac{\partial f}{\partial t}$ at $t=0$

without chain rules : $x = \sin(2t)$, $y = \cos(t)$, $z = x^2 y + 3xy^4$

$\Rightarrow$

$$f(x(t), y(t)) = (\sin(2t))^2 \cos(t) + 3\sin(2t)\cos^4(t)$$

$$\frac{\partial f}{\partial t} = 2\sin(2t)\cdot\cos(2t)\cdot 2\cos(t) + \sin^2(2t)(-\sin(t)) + 3\cos(2t)\cdot 2\cos^4(t) + 3\sin(2t)\,4\cos^3(t)\cdot(-\sin(t)) \Rightarrow 6$$

with chain rules $x = \sin(2t)$, $x(0) = 0$; $y(t) = \cos(t)$, $y(0) = 1$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \quad (2xy + 3y^4)\,2\cos(2t) + \frac{\partial f}{\partial y}\left(\cancel{-\sin(t)}\right) \to 0 \text{ at } 0$$

$x(0) = 0$ $\qquad = 3\cdot 2 = \underline{\underline{6}}$

$y(0) = 1$

Panel 4

Chain Rule is useful for Implicit Differentiation

Suppose $F(x,y) = 0$ is a function defining x and y implicitly, or more precisely, defines $\underline{y = y(x)}$ implicitly.

or $x = x(y)$

$F(x,y) = 0$ Want: $\frac{\partial y}{\partial x}$

$$\frac{\partial}{\partial x} F(x,y) = \frac{\partial}{\partial x} 0 = 0$$

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x} = F_x \cdot 1 + F_y\, y' = 0$$

$$\Rightarrow y' = -\frac{F_x}{F_y}$$

Panel 5

Ex: Find y' if $x^3+y^3=6xy \Leftrightarrow F(x,y)=x^3+y^3-6xy=0$

$x^3+y^3=6xy \quad \Big| \frac{\partial}{\partial x}$

$3x^2+3y^2\cdot\frac{\partial y}{\partial x}=6y+6x\cdot\frac{\partial y}{\partial x} \Rightarrow y'(3y^2-6x)=3x^2-6y$

$y'=\frac{3x^2-6y}{-(3y^2-6x)}=-\frac{F_x}{F_y}$

Ex:

$x^3+y^3=6xy$

$F=x^3+y^3-6xy=0$

$y'=-\frac{F_x}{F_y}$

Panel 6

Technically, not every equation $F(x,y,z)=0$ defines z as an implicitly diff'ble function:

Implicit Function Theorem

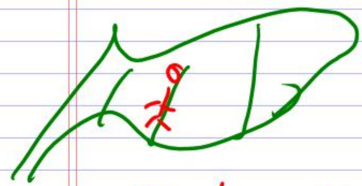
Suppose F is defined in a sphere around (a,b,c) s.t. $F(a,b,c)=0$, $F_z(a,b,c)\neq 0$, and F_x, F_y, F_z are continuous inside that sphere. Then

(a) $F(x,y,z)=0$ defines z as a function of x,y, and

(b) $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are as before!

Panel 7

Directional Derivatives:



f_x is deriv in x-dir

f_y is deriv. in y-dir

Want: slope in direction of $\langle a, b\rangle$

Def: Directional derivative of $f(x,y)$ in the direction $\langle a, b\rangle$

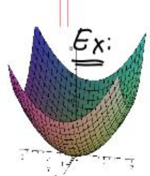
is:

$$D_{\langle a,b\rangle}(f) = \lim_{h\to 0} \frac{f(x+ha,\, y+hb) - f(x,y)}{h}$$

provided that $\langle a, b\rangle$ is a unit vector!

$$D_{\langle 1,0\rangle}(f) = f_x \qquad D_{\langle 0,1\rangle}(f) = f_y$$

Panel 8



Ex: $f(x,y) = x^2 + y^2$ Find directional derivative in direction of $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$ at $(1,1)$

Check $\|\vec{u}\| = 1$ ✓ $\qquad \vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

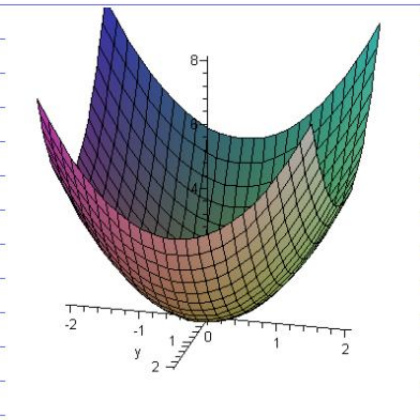
$$D_{\vec{u}}(f) = \lim_{h\to 0} \frac{f\left(x+\frac{h}{\sqrt{2}},\, y+\frac{h}{\sqrt{2}}\right) - f(x,y)}{h} =$$

$$= \lim_{h\to 0} \frac{\left(x+\frac{h}{\sqrt{2}}\right)^2 + \left(y+\frac{h}{\sqrt{2}}\right)^2 - x^2 - y^2}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh/\sqrt{2} + h^2/2 + \cancel{y^2} + 2yh/\sqrt{2} + h^2/2 - \cancel{x^2} - \cancel{y^2}}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(\sqrt{2}\,x + \frac{h}{2} + \sqrt{2}\,y + \frac{h}{2}\right)}{\cancel{h}} = \underline{\sqrt{2}\,(x+y)}$$

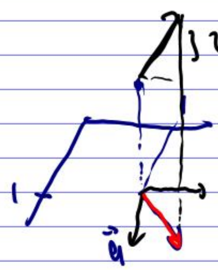
Panel 9



$$f(x,y) = x^2+y^2$$

$$D_{\vec{u}}(f)\Big|_{(1,1)} = \sqrt{2}(x+y)\Big|_{(1,1)} =$$

$$= 2\sqrt{2}$$



$$D_i(f) = f_x = 2x\Big|_{(1,1)} = 2$$

$$D_j(f) = f_y = 2y\Big|_{(1,1)} = 2$$

Panel 10

<u>Thm:</u> Suppose $\vec{u} = \langle a,b \rangle$ is a unit vector. Then

$$D_{\vec{u}} f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\, b = \langle f_x, f_y \rangle \cdot \langle a,b \rangle$$

<u>Ex:</u> $f(x,y) = x^2+y^2$, $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1 \rangle$. Find $D_{\vec{u}} f$ at $(1,1)$

$$D_{\vec{u}}(f) = \langle f_x, f_y \rangle \cdot \left\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \right\rangle =$$

$$= \langle 2x, 2y \rangle \cdot \left\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \right\rangle = \frac{2}{\sqrt{2}}x + \frac{2}{\sqrt{2}}y =$$

$$= \sqrt{2}(x+y)$$

at $(1,1)$: $\underline{\sqrt{2}\cdot 2}$

Panel 11

Gradient: If $f(x,y)$ is a function of 2 variables then the gradient of f is:

$$\text{grad}(f) = \nabla f = \langle f_x, f_y \rangle$$

is a vector with components f_x and f_y

Note: $D_{\vec{u}} f = \langle f_x, f_y \rangle \cdot \vec{u} = \nabla f \cdot \vec{u}$

Works in $\mathbb{R}^2, \mathbb{R}^3, \ldots$

Ex: $f(x,y,z) = x\sin(yz)$, find ∇f

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle \sin(yz), x \cdot z\cos(yz), xy\cos(yz) \rangle$$

Panel 12

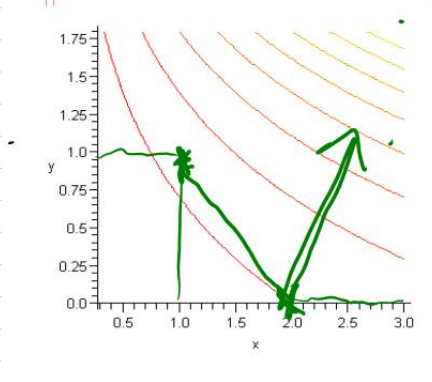
Note: $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$$\Rightarrow |D_{\vec{u}} f| = |\nabla f \cdot \vec{u}| = \|\nabla f\| \cdot \|u\| \cdot \cos(\theta) = \underbrace{\|\nabla f\|}_{\text{fixed}} \cdot \cos(\theta)$$

Useful Theorem: The max. value of $|D_{\vec{u}} f|$ is achieved if $\vec{u}$ points in the direction of ∇f. The maximum value is $\|\nabla f\|$.

Panel 13

Ex: $f(x,y) = xe^y$ Then find rate of change at $P(2,0)$ in direction from P to $Q(1,1)$. In what direction does f have max. rate of change, and what is it?



$\vec{PQ} = \langle -1, 1 \rangle \sim \vec{u} = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle$

$D_{\vec{u}}(f) = (\nabla f) \cdot (\vec{PQ}) =$

$\langle \underbrace{e^y}_{f_x}, \underbrace{xe^y}_{f_y} \rangle \cdot \langle -1, 1 \rangle \frac{1}{\sqrt{2}} =$

$(-e^y + xe^y) \frac{1}{\sqrt{2}}$ at $P(2,0)$

Steepest increase in

$\| \nabla f \|_P = \| \langle e^y, xe^y \rangle \| = \| \langle 1, 2 \rangle \| = \sqrt{5}$ $\quad (-1+2) \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sim 0.7$

Panel 14

Properties of Gradient

- The gradient is a vector $\langle f_x, f_y, f_z \rangle$
- Gradient is perp. to level curves
- Gradient points in direction of steepest increase
- $\| \nabla f \|$ is the is the max increase.

Ex: Find ∇f if $f(x,y,z) = \ln(xy^2z^3)$ HW

Find max. rate of change for $f(x,y,z) = \ln(xy^2z^3)$ at $P(1,1)$