Panel 1

To tackle limits in $\mathbb{R}^2$ (or $\mathbb{R}^3$)

① Try the obvious (substitute + hope for the best)

② Try different approach paths. Commonly used:

$x=0$
$y=0$
$x=y$
$x=y^2$
$y=x^2$

if any answers are different, DONE

③ Try to prove that the limit is the common number found in step ②

!

Panel 2

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ DNE

$x=0, y\to 0$: $\lim_{y\to 0} \frac{0}{y^2} = 0$

$y=0, x\to 0$: $\lim \frac{x^2}{x^2} = 1$

Ex: $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$

$x=0, y\to 0$: $\lim = 0$

$y=0, x\to 0$: $\lim = 0$

$x=y$: $\lim_{x\to 0} \frac{2x^3}{x^4+x^2} = \lim_{x\to 0} \frac{2x^{\cancel{3}}}{\cancel{x^2}(x^2+1)} = 0$

Ex: $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

$x=0$: $\lim = 0$

$y=0$: $\lim = 0$

$x=y$: $\lim_{x\to 0} \frac{3x^3}{2x^2} = 0$

Panel 3

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$$

$x=0$: $\lim = 0$

$y=0$: $\lim = 0$

$x=y$: $\lim_{x\to0} \frac{3x^3}{2x^2} = 0$

$y=x^2$: $\lim_{x\to0} \frac{3x^2x^2}{x^2+x^4} = \lim_{x\to0} \frac{3x^{\cancel{4}\,2}}{\cancel{x^2}(1+x^2)} = 0$

$x=y^2$: $\lim_{y\to0} \frac{3y^4y}{y^4+y^2} = \lim_{y\to0} \frac{3y^5}{y^2(y^2+1)} = 0$

Maybe the limit exists!?!

Panel 4

Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

Take $\varepsilon>0$. Want to find $\delta>0$ s.t.

$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$ if $\|(x,y)-(0,0)\| < \delta$

$\sqrt{x^2+y^2} < \delta$

trick!

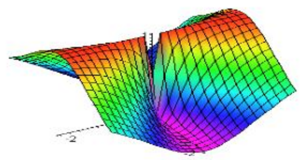
Note: $x^2 \le x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} \le 1$ $\qquad \boxed{\Rightarrow |f(x,y)-0| < \varepsilon}$

$\Rightarrow \frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2}$

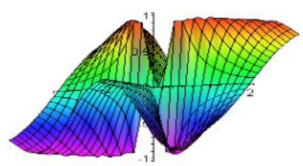
Proof: Pick $\varepsilon>0$, let $\delta = \frac{\varepsilon}{3}$!!!! If $\sqrt{x^2+y^2} < \delta = \varepsilon/3$

Panel 5

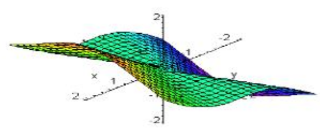
$$f(x,y) = \frac{x^2}{x^2+y^2}$$



$$f(x,y) = \frac{2x^2 y}{x^4+y^2}$$



$$f(x,y) = \frac{x^3}{x^2+y^2}$$



Panel 6

<u>Continuity</u>

As usual, continuity is just

<u>Def.</u> $f(x,y)$ is continuous at (x_0,y_0) if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(x_0,y_0)$$

<u>Ex:</u>

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ \cancel{2}\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$f(0,0) = 2$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$ Not cont.

Now it is!!

Panel 7

Derivatives:

If $f(x,y)$ is a function of 2 variables, define

$$\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=\frac{\partial f}{\partial x}=f_x$$ partial deriv. with respect to x

$$\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}=\frac{\partial f}{\partial y}=f_y$$ partial w.r.t. y

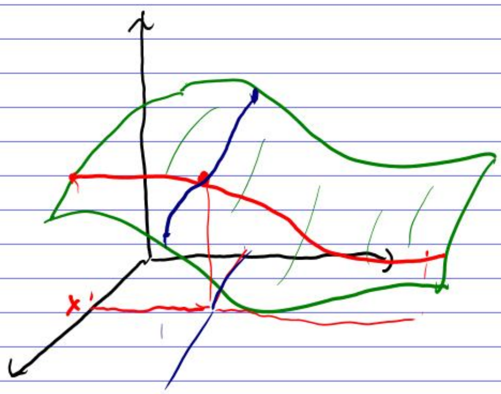
Ex: $f(x,y)=x^2y+y^2 \Rightarrow \frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=$

$$=\lim_{h\to 0}\frac{\left((x+h)^2y+\cancel{y^2}\right)-\left(x^2y+\cancel{y^2}\right)}{h}=\lim_{h\to 0}\frac{\cancel{x^2y}+2xhy+h^2y-\cancel{x^2y}}{h}=$$

$$=\lim_{h\to 0}\frac{h(2xy+hy)}{h}=2xy$$

$$\frac{\partial f}{\partial y}=x^2+2y$$

Panel 8



$\frac{\partial f}{\partial y}$ gives slope of tangent to curve $f(x,y)$, x-fixed, i.e. in the y-direction

$\frac{\partial f}{\partial x}$ is slope of curve $f(x,y)$, y fixed, i.e. in the x-direction!

Panel 9

Ex: $f(x,y) = x^3 + x^2y^3 - 2y^2$. Find

$f_x(2,1)$: $f_x(x,y) = 3x^2 + 2xy^3$

$\rightarrow f_x(2,1) = 12 + 4 = 16$

$f_y(2,1)$: $f_y(x,y) = 3x^2y^2 - 4y$

$f_y(2,1) = \ldots$

Panel 10

3D Example: $f(x,y,z) = xz\,e^{x^2+y^2}$. Find

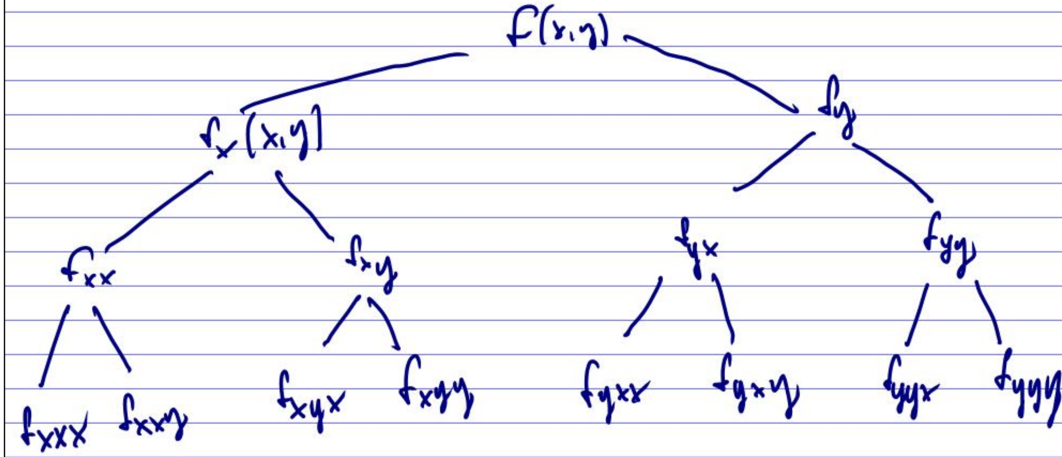
$f_x(x,y,z) = z e^{x^2+y^2} + xz2x\,e^{x^2+y^2}$

$f_y(x,y,z) = xz2y e^{x^2+y^2}$

$f_z(x,y,z) = xe^{x^2+y^2}$

Panel 11

Of course we can also take higher-order partial derivatives. Let $f(x,y)$ be a function



Usually, many of these are the same!

Panel 12

$\underline{Ex:}$ $f(x,y) = \underline{x^3} + \underline{x^2y^3} - \underline{2y^2}$

$f_x(x,y) = \underline{3x^2} + \underline{2xy^3}$

$f_y(x,y) = 3x^2y^2 - 4y$

$f_{xx}(x,y) = 6x + 2y^3$

$f_{xy}(x,y) = 6xy^2$

$f_{yx}(x,y) = 6xy^2$

$f_{yy}(x,y) = 6x^2y - 4$

$\underline{Note:}$

$f_{xy} = f_{yx}$

(usually!)