Panel 1

Functions of Several Variables

I. Know. $f: \mathbb{R} \to \mathbb{R}$ e.g. $f(x) = x^2$

$r: \mathbb{R} \to \mathbb{R}^2$ e.g. $r(t) = \langle t^2, t^3 \rangle$

$r: \mathbb{R} \to \mathbb{R}^3$ e.g. $r(t) = \langle \cos t, \sin t, t \rangle$

II. $f: \mathbb{R}^2 \to \mathbb{R}$ $g: \mathbb{R}^3 \to \mathbb{R}$

e.g. $f(x,y) = x+y$ e.g. $g(x,y,z) = x^2 y^2 z^2$

III. $F: \mathbb{R}^2 \to \mathbb{R}^2$, e.g. $F(x,y) = \langle xy, x^2y^2 \rangle$

$G: \mathbb{R}^2 \to \mathbb{R}^3$, e.g. $G: (x,y) = \langle xy, x^2y, xy^2 \rangle$

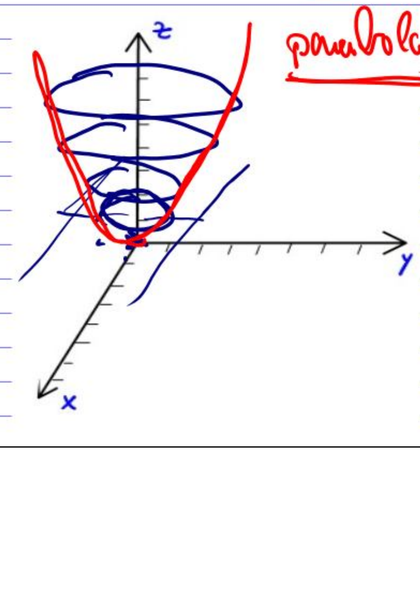
Panel 2

Def: A function of 2 variables is a rule that assigns to every pair (x,y) in a set $D \subset \mathbb{R}^2$ exactly one number $z = f(x,y)$

Cuts: fix $x=c$ or $y=c$ or $z=c$

$x=0$: $\Rightarrow z = y^2$

Ex: $f(x,y) = x^2 + y^2 = z$



paraboloid

(x,y)	z
$(0,0)$	0
$(1,0)$	1
$(0,-1)$	1
$(1,1)$	2
$(-1,-1)$	2
⋮	⋮

Note: $x^2+y^2=0$

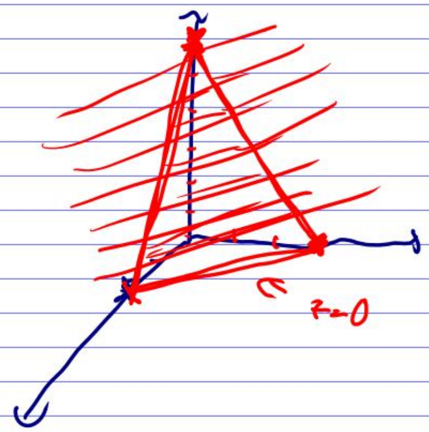
$x^2+y^2=1$, $z=1$

Panel 3

Ex: $f(x,y) = 6 - 3x - 2y$

$z = 6 - 3x - 2y$, i.e.

$3x + 2y + z = 6$



$x=0$: $2y + z = 6$

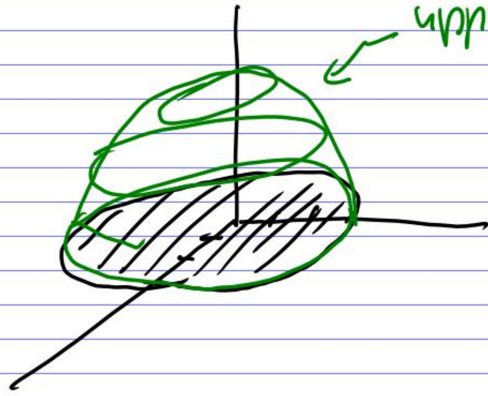
$y=0$: $3x + z = 6$

Panel 4

Ex: $f(x,y) = \sqrt{9 - x^2 - y^2}$

Domain: $9 - x^2 - y^2 \geq 0 \iff 9 \geq x^2 + y^2$

upper half sphere!



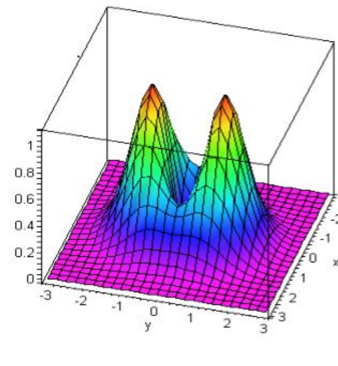
$z = \sqrt{9 - x^2 - y^2}$

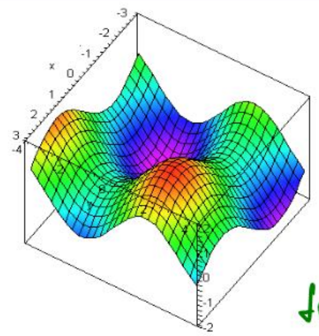
$z^2 = 9 - x^2 - y^2$

$9 = x^2 + y^2 + z^2$

Panel 5

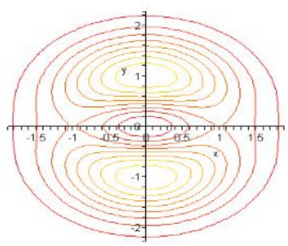
$f(x,y) = (x^2+3y^2)\,e^{-x^2-y^2}$ $f(x,y) = \sin(x) + \sin(y)$

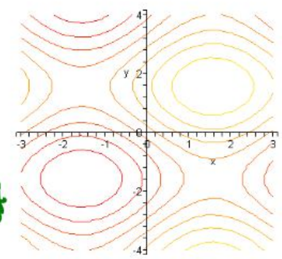




topological maps

contour plots: (x,y) s.t. $f(x,y) = \text{const}$





Panel 6

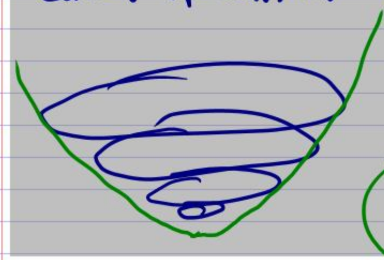
Of course I used Maple to generate these plots

```
> plot3d((x^2+3*y^2)*exp(-x^2-y^2),x=-3..3,y=-4..4);

> plot3d(sin(x)+sin(y),x=-3..3,y=-4..4);

> with(plots);

> contourplot((x^2+3*y^2)*exp(-x^2-y^2),x=-3..3,y=-4..4);

> contourplot(sin(x)+sin(y),x=-3..3,y=-4..4);

>
```

Panel 7

Ex: Level curves of $h(x,y) = 4x^2+y^2$

Curves of constant height. $4x^2+y^2=1$



ellipses

contour plot

Ex: Level curves for $f(x,y) = 6-3x-2y$

plane

$6-3x-2y=1,2,3,4,\ldots$

lines

Panel 8

Limits: The limit of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L is written as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

Given any $\varepsilon > 0$ there is $\delta > 0$ such that:

whenever $\|(x,y)-(x_0,y_0)\| < \delta$ then

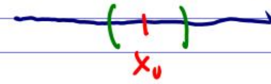
$$|f(x,y) - L| < \varepsilon$$

$$\sqrt{(x-x_0)^2-(y-y_0)^2} < \delta \iff (x-x_0)^2+(y-y_0)^2 \le \delta^2$$

Panel 9

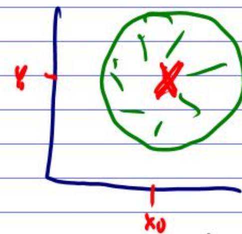
In $\mathbb{R}$: $\lim_{x \to x_0} f(x)$

2 sides $\Leftrightarrow$ left + right limit



domain

In $\mathbb{R}^2$: $\lim_{(x,y) \to (x_0,y_0)} f(x,y)$



Bummer: Inf. many ways to get close to (x_0, y_0) in $\mathbb{R}^2$.
If they all agreed that would be the limit!!!

hard to show $\Rightarrow$ that a limit exists!

Panel 10

Hints for finding limits in $\mathbb{R}^2$:

$\Rightarrow$ if C_1 is a path to (x_0,y_0) and $f(x,y) \to L_1$ on C_1

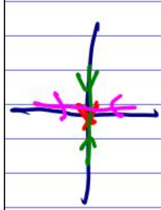
$\Rightarrow$ if C_2 is a path to (x_0,y_0) and $f(x,y) \to L_2$ on C_2

If $L_1 \neq L_2$ $\Rightarrow$ limit DNE

If $L_1 = L_2$ means nothing, maybe, tips, field

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist

Fix $x=0$, $y \to 0$: $\lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$ different

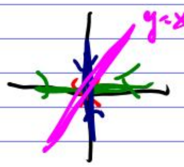


Fix $y=0$, $x \to 0$: $\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$

Panel 11

Ex: $\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2} = \frac{1}{2}$

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$: DNE



$x=x,\ y=0.$ $\lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$ tells me nada, zip

$x=0,\ y\to 0$ $\lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0$

$y=x\to 0$ $\lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$

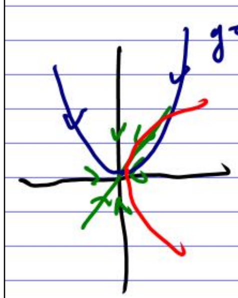
Panel 12

Ex: Find $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ if it exists DNE (H4!)

$x=0,\ y\to 0$: $\lim = 0$

$y=0,\ x\to 0$: $\lim = 0$

$y=x\to 0.$ $\lim_{x\to 0} \frac{x^3}{x^2+x^4} = \lim \frac{x^3}{\cancel{x^2}(1+x^2)} = \lim_{x\to 0} \frac{x}{1+x^2} = 0$



$y=x^2\to 0.$ $\lim_{x\to 0} \frac{x^5}{x^2+x^8} = 0$

$x=y^2$: $\lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2}$ different

Panel 13

Ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

HW – try, but not too hard

$x=0$

$y=0$

$x=y$

$x=y^2$

$y=x^2$