Panel 1

Summary

$\vec{r}(t) = (x(t), y(t), z(t))$

$\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

$\vec{T}(t) = r'(t)/\|r'\|$ (easy)

$\vec{N}(t) = T'(t)/\|T'\|$ (hard)

$\vec{B}(t) = T \times N$

$\kappa(t) = \dfrac{\|T'\|}{\|r'\|} = \dfrac{\|r'(t) \times r''(t)\|}{\|r'\|^3}$ curvature

Panel 2

Ex: Find the osculating plane of

$\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$ at $P(0, 1, \pi/2)$.

That plane that supports curve the best.

T and N are in the plane.

Thus B defines the plane!

$\vec{T} = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle$

$\vec{N} = \langle -\cos(t), -\sin(t), 0\rangle$

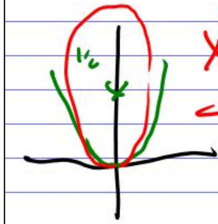
$\vec{B} = \frac{1}{\sqrt{2}}\langle \sin(t), -\cos(t), 1\rangle = T \times N$ (last time)

$t = \frac{\pi}{2} \Rightarrow P \quad \vec{B} = \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle \Rightarrow \vec{n} = \langle 1, 0, 1\rangle$

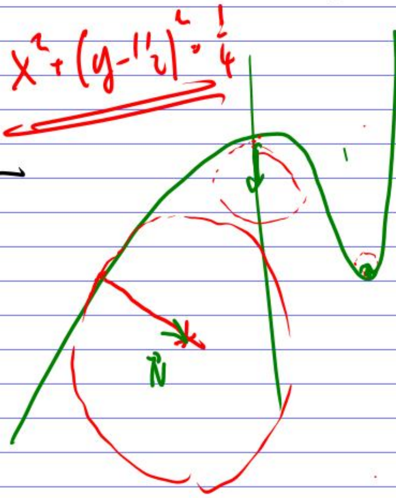
$x + z = \pi/2$

Panel 3

Ex: Find the osculating circle to $y = x^2$ at $(0,0)$. $(t=0)$



$x^2 + (y - 1/2)^2 = \frac{1}{4}$



radius: $1/\kappa$

Along $\vec{N}$

$y = x^2$

$r(t) = \langle t, t^2, 0 \rangle = \langle 0,0,0 \rangle$

$r'(t) = \langle 1, 2t, 0 \rangle = \langle 1,0,0 \rangle$

$r''(t) = \langle 0, 2, 0 \rangle = \langle 0,2,0 \rangle$

$r' \times r'' = \langle 0,0,2 \rangle \Rightarrow \kappa = \frac{2}{1^3} = 2$

Note: Any function $y = f(x)$ can be written in parametric form: $\vec{r}(t) = \langle t, f(t), 0 \rangle$

Panel 4

$\vec{r} = \langle a\cos(t), a\sin(t), 5t \rangle$

$r' = \langle -a\sin(t), a\cos(t), 5 \rangle$

$r'' = \langle -a\cos(t), -a\sin(t), 0 \rangle$

$r' \times r'' = \langle 5a\sin(t), -5a\cos(t), a^2\sin^2(t) + a^2\cos^2(t) \rangle$

$= \langle 5a\sin(t), -5a\cos(t), a^2 \rangle$

$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\sqrt{a^2 5^2 + a^4}}{\left(\sqrt{a^2 + 5^2}\right)^3}$ a constant

Panel 5

Motion in Space

Suppose $\vec{r}(t)$ represents the motion of a particle in space depending on time t.

$\vec{r}(t)$ is motion, i.e. location in space / time

$\vec{r}\,'(t)$ is velocity: $v(t) = r'(t)$

$\|r'(t)\|$ is speed. $s = \|v(t)\|$

r'' is acceleration: $a(t) = v'(t) = r''(t)$

Panel 6

Ex: Suppose the path of a particle at time t is $\vec{r}(t) = \langle t^3, t^2 \rangle$. Find velocity, speed, and acceleration when $t = 1$. Illustrate.

$v(t) = \langle 3t^2, 2t \rangle \Rightarrow v(1) = \langle 3, 2 \rangle$

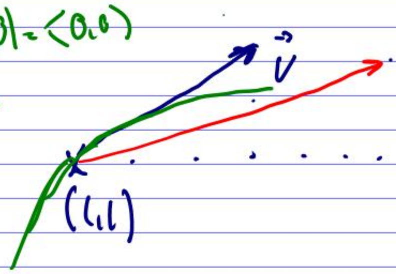
$s(t) = \sqrt{9t^4 + 4t^2} \Rightarrow s(1) = \sqrt{13}$

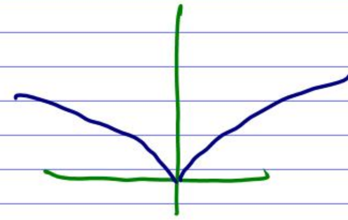
$a(t) = \langle 6t, 2 \rangle \Rightarrow a(1) = \langle 6, 2 \rangle$

Not smooth: $r'(0) = \langle 0, 0 \rangle$

$\Rightarrow$ has a corner at $(0,0)$

at $(1,1)$, speeding up





Panel 7

Ex: A particle starts at P(1, 0, 0) with initial velocity $\langle 1, -1, 1 \rangle$. The acceleration is $\vec{a}(t) = \langle 4t, 6t, 1 \rangle$
Find velocity, speed, and position.

$$a(t) = \langle 4t, 6t, 1 \rangle$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 2t^2, 3t^2, t \rangle + v_0 \Rightarrow v(0) = \langle 1, -1, 1 \rangle$$

$$\vec{v}(t) = \langle 2t^2+1, 3t^2-1, t+1 \rangle$$

$$r(t) = \langle \tfrac{2}{3}t^3 + t, t^3 - t, \tfrac{1}{2}t^2 + t \rangle + r_0, \quad \vec{r}(0) = \langle 1, 0, 0 \rangle$$

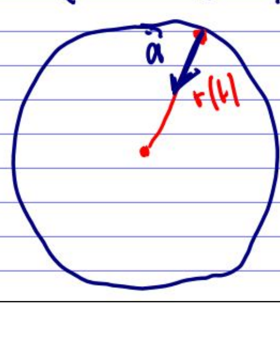
Panel 8

Ex: An object with mass m moves in a circle with constant angular speed ω. Find the force acting on the object and illustrate.

$$r(t) = \langle 1\cos(\omega t), 1\sin(\omega t) \rangle$$

$$v(t) = \langle -\omega \sin(\omega t), \omega\cos(\omega t) \rangle, \quad s = \|v\| = \omega,$$

$$a(t) = \langle -\omega^2 \cos(\omega t), -\omega^2 \sin(\omega t) \rangle = -\omega^2\, r(t)$$



$$\vec{F} = m\vec{a}$$

Panel 9

Application of Motion

A baseball is hit 3 feet above ground at 100 feet per second and at an angle of Pi/4 with respect to the ground. Find the maximum height reached by the baseball. Will it clear a 10-foot high fence located 300 feet from home base?

HW

$\vec{a}(t) = \langle 0, -g \rangle$

$\vec{v}(t) = \_\_$

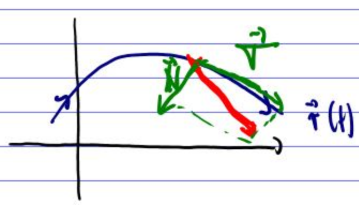
$\vec{r}(t) = \langle x(t), y(t) \rangle$ find $y'(t) = 0$, Find $y(t_0)$

Solve $x(t) = 300$ for t: $x(t_1) = 300$.

Find $y(t_1) > 10$ or not?

Panel 10

Tangential and Normal Components of Acceleration



acceleration can be divided into:

portion in direction of $\vec{T}$

portion in direction of $\vec{N}$

$\vec{a} = a_T \cdot \vec{T} + a_N \cdot \vec{N}$.

a_T specifies if particle speeds up or slows down

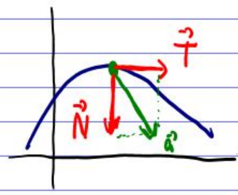
a_N how fast particle changes direction

Panel 11

Formulas : $\vec{a} = a_T \vec{T} + a_N \vec{N}$, where

tang. comp. $a_T = \dfrac{\vec{v}\cdot\vec{a}}{s}$

normal comp $a_N = \dfrac{\|v \times a\|}{s}$



$\vec{a} = a_t \cdot \vec{T} + a_N \vec{N}$

Panel 12

Quiz 4

Suppose $\vec{r}(t) = \langle t^2, 2, t \rangle$ is a vector-valued function (aka space curve), representing the position of a particle. Find the following:

1. The velocity at $P(0,0,0)$
2. The speed at $P(0,0,0)$
3. The acceleration at $P(0,0,0)$
4. The unit tangent $\vec{T}(t)$ at $P(0,0,0)$
5. The unit normal vector $\vec{N}(t)$ at $P(0,0,0)$
6. The bi-normal vector $\vec{B}(t)$ at $P(0,0,0)$
7. The curvature k at $P(0,0,0)$
8. The tangential component of the acceleration a_T at $P(0,0,0)$
9. The normal component of the acceleration a_N at $P(0,0,0)$
10. The osculating plane at $P(0,0,0)$
11. The osculating circle at $P(0,0,0)$