Panel 1

Last Time

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ a space curve

Length: $d = \int_a^b \|r'(t)\| \, dt = \int_a^b \sqrt{(x')^2 + (y')^2 + (z')^2} \, dt$

Unit Tangent $T(t) = r'(t) / \|r'(t)\|$

Unit Normal $N(t) = T'(t) / \|T'(t)\|$ (Hard!)

Unit Binormal: $B(t) = T(t) \times N(t)$

Panel 2

Theorem: $r(t)$ space curve s.t. $\|r(t)\| = 1$. Then
$r(t) \cdot r'(t) = 0$

$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$

Why: $\|r(t)\| = 1$

$\|r(t)\|^2 = 1$

$r(t) \cdot r(t) = 1 \quad \Big| \frac{d}{dt}$

$r'(t) \cdot r(t) + r(t) \cdot r'(t) = 0$

$2 r \cdot r' = 0$

$\Rightarrow r \cdot r' = 0$, i.e. are perp.

$\Rightarrow N(t)$ is therefore perp. to T because

Panel 3

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit tangent, unit normal and binormal vectors.

$$r'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$\|r'(t)\| = \sqrt{\sin^2 + \cos^2 + 1} = \sqrt{2}$$

$$\Rightarrow T = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle.$$

is $N \perp T$?

$$T' = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle \quad , \|T'\| = \frac{1}{\sqrt{2}} \cdot 1$$

$$\Rightarrow N = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \langle -\cos(t), -\sin(t), 0 \rangle = \langle -\cos(t), -\sin(t), 0 \rangle$$

Panel 4

$$B(t) = T \times N = \begin{vmatrix} i & j & k \\ -\frac{1}{\sqrt{2}} \sin(t) & \frac{1}{\sqrt{2}} \cos(t) & 1/\sqrt{2} \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} =$$

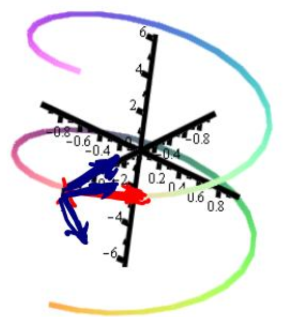
$$= \langle \sin(t), \cos(t), \frac{1}{\sqrt{2}} \sin^2(t) + \frac{1}{\sqrt{2}} \cos^2(t) \rangle =$$

$$= \langle \frac{1}{\sqrt{2}} \sin(t), -\frac{1}{\sqrt{2}} \cos(t), \frac{1}{\sqrt{2}} \rangle =$$

$$= \underline{\underline{\frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle}}$$

Panel 5

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find unit tangent, unit normal and binormal vectors at $t=0$. Visualize!



$$T = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$$

$$N = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$B = \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle$$

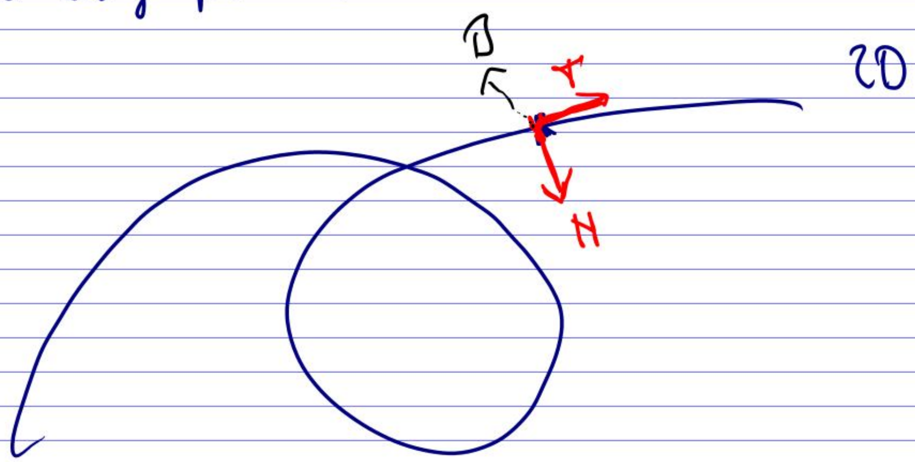
at $t=0$: $$T = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$$

$$N = \langle -1, 0, 0 \rangle$$

$$B = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle$$

Panel 6

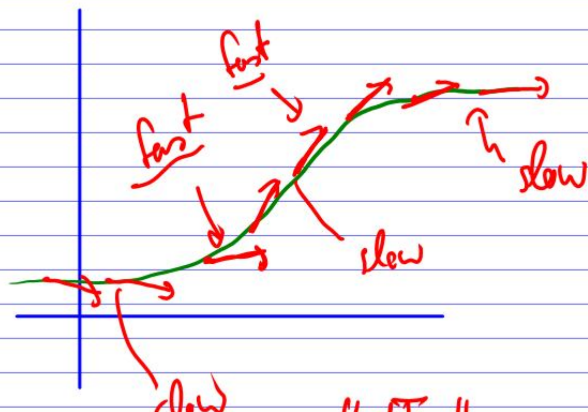
T, N, and B form a local coordinate system at every point t.



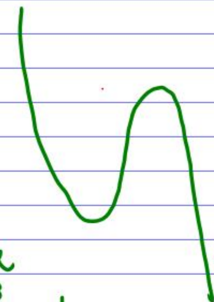
Panel 7

Unit tangent give direction of change, normalized

⇒ Want to find rate of change of tangent



$\left\| \frac{dT}{dt} \right\|$ measure curvature!



Panel 8

We can now measure direction of curve (derivative) and length (arc length). Next we measure

Def: The curvature $\kappa = \cancel{\left\| \frac{dT}{dt} \right\|} = \frac{\|T'(t)\|}{\|r'(t)\|}$

measures how quickly curvature changes!

Panel 9

Ex: What is the curvature of a circle of radius r?

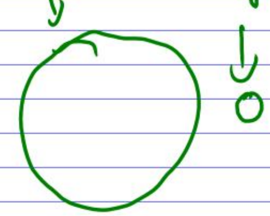
$r(t) = \langle r\cdot\cos(t), r\cdot\sin(t)\rangle$

$\kappa = \frac{\|T'\|}{\|r'\|}$ $r'(t) = \langle -r\sin(t), r\cos(t)\rangle$

$\|r'(t)\| = \sqrt{r^2\sin^2 + r^2\cos^2} = \sqrt{r^2} = r$

$\Rightarrow T(t) = \frac{r'}{\|r'\|} = \langle -\sin(t), \cos(t)\rangle$

big r, small κ small r big κ



$\Rightarrow T'(t) = \langle -\cos(t), -\sin(t)\rangle$

$\kappa = \frac{\|T'\|}{\|r'\|} = \frac{1}{r}$ const.

Panel 10

Ex: Find curvature of $r(t) = \langle t, t^2\rangle$ $\kappa = \frac{\|T'\|}{\|r'\|}$

$r'(t) = \langle 1, 2t\rangle$, $\|r'(t)\| = \sqrt{1+4t^2}$

$T(t) = \frac{1}{\sqrt{1+4t^2}}\langle 1,2t\rangle = (1+4t^2)^{-1/2}\langle 1,2t\rangle$

$T'(t) = -\frac{1}{2}(1+4t^2)^{-3/2}\cdot 8t\langle 1,2t\rangle + (1+4t^2)^{-1/2}\langle 0,2\rangle$

$= (1+4t^2)^{-1/2}\left(-(1+4t^2)^{-1}\cdot 4t\langle 1,2t\rangle + \langle 0,2\rangle\right) =$

$= (1+4t^2)^{-1/2}\left(-\frac{4t}{1+4t^2}\langle 1,2t\rangle + \langle 0,2\rangle\right)$

Panel 11

$$= \left((1+4t^2)^{-1/2}\left(\frac{-4t}{1+4t^2}\langle 1,2t\rangle + \langle 0,2\rangle\right)\right.$$

$$\chi = \frac{\|T'\|}{\|r'\|}$$

$$= (1+4t^2)^{-1/2}\left\langle \frac{-4t}{1+4t^2}, \frac{8t^2}{1+4t^2}+2 \right\rangle =$$

$$= (1+4t^2)^{-1/2}\left(\frac{-4t}{1+4t^2}, \frac{-8t^2+2+8t^2}{1+4t^2}\right) =$$

$$= (1+4t^2)^{-3/2}\langle -4t, 2\rangle = T'$$

$$= \frac{2}{1+4t^2}$$

$$\|T'\| = (1+4t^2)^{-3/2}\sqrt{16t^2+4} = 2(1+4t^2)^{-3/2}(4t^2+1) =$$

Panel 12

$$\|T'\| = \frac{2}{1+4t^2}$$

$$r'(t) = \langle 1, 2t\rangle, \quad \|r'\| = (1+4t^2)^{1/2}$$

$$\Rightarrow \chi = \frac{\|T'\|}{\|r'\|} = \frac{2}{(1+4t^2)^{3/2}}$$

Ex: $r(t) = \langle t, t^2, t^3\rangle$

$r'(t) = \langle 1, 2t, 3t^2\rangle$, $\|r'\| = \sqrt{1+4t^2+9t^4}$

$$T = \frac{1}{\sqrt{1+4t^2+9t^4}}\langle 1, 2t, 3t^2\rangle \qquad T' = ?\ \text{Groups}$$

Panel 13

Ex: Find curvature of $r(t) = \langle t, t^2, t^3 \rangle$

too hard

Panel 14

Thm. The curvature of $\vec{r}(t)$ is

$$\kappa(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{\|\langle 0,0,2\rangle\|}{\left(\sqrt{1+4t^2}\right)^3} = \frac{2}{(1+4t^2)^{3/2}}$$

Ex: $\vec{r}(t) = \langle t, t^2 \rangle \in \mathbb{R}^2$ can be embedded into $\mathbb{R}^3$

$= \langle t, t^2, 0 \rangle$

$r' = \langle 1, 2t, 0 \rangle$

$r'' = \langle 0, 2, 0 \rangle$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

Panel 15

Summary

$r(t)$ = Space curve — space curve

$r'(t) = \langle x'(t), y'(t), z'(t) \rangle$ — tangent vector

$T(t) = r'(t) / \|r'\|$ — unit tangent

$N(t) = T'(t) / \|T'\|$ — principle normal

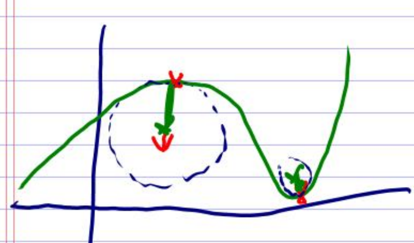
$B(t) = T \times N$ — binormal

$\kappa = \dfrac{\|T'(t)\|}{\|r'\|} = \dfrac{\|r' \times r''\|}{\|r'\|^3}$ — curvature

Panel 16

Def: The plane determined by T and N is called osculating plane or supporting plane. (Latin: osculum = kiss)

Def: The circle in the osculating plane with radius $r = 1/\kappa$ is called the osculating circle.



Draw osculating circle at both points

Panel 17

Motion in Space

Suppose $\vec{r}(t)$ represents the