Panel 1

Last time

Coordinate system in $\mathbb{R}^3$

Distance formula

3D Objects ✓

Vectors

1

Panel 2

Calc 3 - Quiz #1

① Find the distance between $P(-1,2,0)$ and $Q(2,1,1)$.

✓

② Find radius of sphere $x^2+y^2+z^2-6x+4y-2z=11$

$r=5$

2

Panel 3

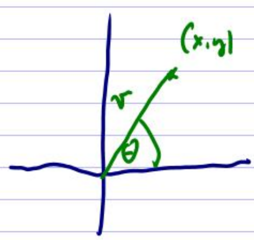
(3) Describe 3D object given by $x^2 + z^2 = 4$

(4) Find a vector in direction $\langle -3, 4, 5 \rangle$ with length 2.

3

Panel 4

Other ways to describe vectors: Find a vector of length 2 polar that makes an angle of $\pi/3$ with positive x-axis.



Polar (2D)

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$x = 2\cos(\frac{\pi}{3}) = 2 \cdot \frac{1}{2}$ $\iff$ $r^2 = x^2 + y^2$

$y = 2\sin(\frac{\pi}{3}) = 2\frac{\sqrt{3}}{2}$ $\theta = \arctan(\frac{x}{y})$

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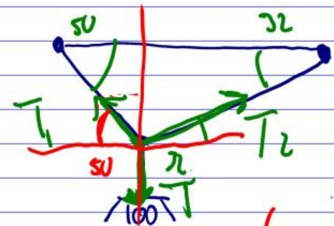
Panel 5

What are vectors good for: a 100 lb weight hangs from two wires as shown. Find the forces T_1 and T_2 acting on the wires and their magnitudes.



created out

$T_1 = \|T_1\| \langle -\cos(50), \sin(50) \rangle$

$T_2 = \|T_2\| \langle \cos(32), \sin(32) \rangle$

2 equations

$T = \langle 0, 100 \rangle$

$T_1 + T_2 = T = \langle 0, 100 \rangle$

$\|T_2\| = \|T_1\| \frac{\cos(50)}{\cos(32)}$

$-\|T_1\| \cos(50) + \|T_2\| \cos(32) = 0$

$\|T_1\| \sin(50) + \|T_2\| \sin(32) = 100$ $\Rightarrow$

$\|T_1\| = 85.64$

$\|T_2\| = 64.91$

Panel 6

Know how to add / subtract vectors.

$\Rightarrow$ How to multiply?

$\vec{v} + \vec{w} = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle = \langle v_1 + w_1, v_2 + w_2 \rangle$

Try: $\vec{v} \cdot \vec{w} = \langle v_1 w_1, v_2 w_2 \rangle$????

No good: $\langle 0, 1 \rangle \cdot \langle 5, 0 \rangle = \langle 0, 0 \rangle$ Now $a \cdot b = \vec{0}$ but neither $\vec{a} = \vec{0}$ nor $\vec{b} = \vec{0}$

Dot Product

2D: $\vec{v} \cdot \vec{w} = \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 \cdot w_1 + v_2 w_2$

3D. $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

Panel 7

Examples of Dot Product

① $\langle 3,5\rangle \cdot \langle -1,2\rangle =$ $3(-1) = -3$
$5(2) = 10$
$= 7$

② $\langle 2,3\rangle \cdot \langle -3,2\rangle =$ $-6 + 6 = 0$

③ $\langle 1,-3,4\rangle \cdot \langle 1,5,2\rangle =$ $1 \cdot 1 - 3 \cdot 5 + 4 \cdot 2 = 1 - 15 + 8 = -6$

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Panel 8

Properties of Dot Product

! a) $\vec{a} \cdot \vec{a} = \|a\|^2$

b) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

c) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Proof (a): $\vec{a} = \langle a_1, a_2, a_3\rangle \Rightarrow \vec{a} \cdot \vec{a} = \langle a_1, a_2, a_3\rangle \cdot \langle a_1, a_2, a_3\rangle =$
$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ $\quad = a_1^2 + a_2^2 + a_3^2 = \|\vec{a}\|^2$

Proof (b) Try as HW in 2D
in 3D

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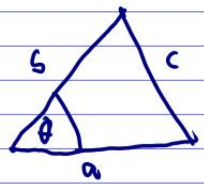
Panel 9

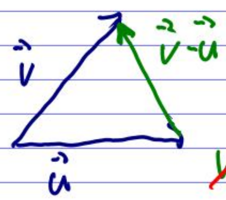
Theorem: If u and v are non-zero vectors in $\mathbb{R}^2$ then

$$\frac{u \cdot v}{\|u\| \cdot \|v\|} = \cos(\theta) \qquad \vec{u} \cdot \vec{v} = \cos(\theta) \cdot \|u\| \cdot \|v\|$$

Law of cosines



$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$



$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos(\theta)$$

$$(v-u)\cdot(v-u) = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos(\theta)$$

$$\cancel{v\cdot v} + \cancel{u\cdot u} + \cancel{2}(u\cdot v) = \cancel{\|u\|^2} + \cancel{\|v\|^2} + \cancel{2}\|u\|\|v\|\cos(\theta)$$

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Panel 10

Ex: Find angle between $u = i - 2j + 2k = \langle 1,-2,2\rangle$ and

a) $v = -3i + 6j + 2k = \langle -3,6,2\rangle$

$u\cdot v = \langle 1,-2,2\rangle \cdot \langle -3,6,2\rangle = -3 - 12 + 4 = -11$

$\|u\| = \sqrt{9} = 3$

$\|v\| = \sqrt{49} = 7$

$\cos(\theta) = \frac{-11}{3\cdot 7} = -\frac{11}{21}$

$\theta = \arccos\left(\frac{-11}{21}\right)$

b) $w = 2i + 7j + 6\vec{k}$

$= \langle 2,7,6\rangle$

$u\cdot w = 2 - 14 + 12 = 0 \quad \Rightarrow \cos(\theta) = 0$

$\Rightarrow \theta = \pi/2$

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Panel 11

Corollary: Two vectors $\vec{v}$ and $\vec{w}$ are perpendicular iff $\vec{v} \cdot \vec{w} = 0$

Ex: Which of the following vectors are perpendicular?

a) $\langle 1, 2, 3 \rangle$ and $\langle -1, -2, -3 \rangle$

b) $\langle 1, 2, 3 \rangle$ and $\langle -1, -3, 2 \rangle$

c) $\langle 1, 2, 3 \rangle$ and $\langle 6, -1, 1 \rangle$

d) $\langle 1, 2, 3 \rangle$ and $\langle 5, -1, -1 \rangle = 0$ ✓

e) $\langle 1, 2, 3 \rangle$ and $\langle 0, -3, 2 \rangle = 0$ ✓

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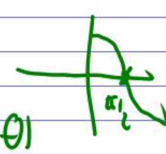
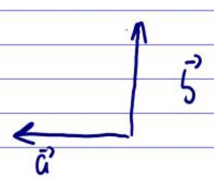
Panel 12

Picture Problems

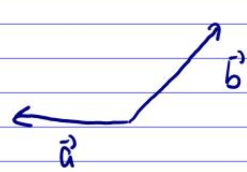
$$\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} = \cos\theta \iff \vec{a} \cdot \vec{b} = \overbrace{\|a\| \, \|b\|}^{+} \cos(\theta)$$



$\vec{a} \cdot \vec{b}$ — positive / zero / negative



$\vec{a} \cdot \vec{b}$ — positive / zero / negative



$\vec{a} \cdot \vec{b}$ — positive / zero / negative

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Panel 13

How many vectors are perp. to $\langle 2,3\rangle$ in $\mathbb{R}^2$?
Find them: $\langle 3,-2\rangle$ and

$\langle -3,2\rangle$ and scalar multiples

How many vectors are perp. to $\langle 1,2,3\rangle$ in $\mathbb{R}^3$.
Find a few of them

$\left.\begin{array}{l}\langle 0,-3,2\rangle \\ \langle 3,0,-1\rangle \\ \langle 2,-1,0\rangle\end{array}\right\}$ and inf. many more!

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Panel 14

Ex: Find the angle that $\vec{a}=\langle 1,2,3\rangle$ makes with the y-axis:

$$\cos(\theta_y)=\frac{\vec{a}\cdot\langle 0,1,0\rangle}{\|a\|\cdot 1}=\frac{\langle a_1,a_2,a_3\rangle\cdot\langle 0,1,0\rangle}{\|a\|} \quad \text{(directional angle)}$$

$$=\frac{a_2}{\|a\|}$$

$\cos(\theta_x)$

HW

$\cos(\theta_z)$

Ex: $\vec{a}=\langle 3,5,7\rangle$
What is θ_y, directional angle of $\vec{a}$ with y-axis?

$$\cos(\theta_y)=\frac{5}{\sqrt{9+25+49}}$$

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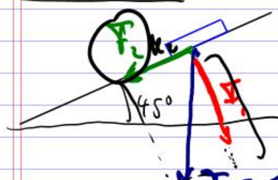
Panel 15

Application: Suppose a 10Kg block is on a 45° incline. What is the force pulling the block in the direction of the incline?



$F = \langle 0, -10 \rangle$

$F = F_1 + F_2$, F_1, F_2 are perp.

$\Rightarrow F = k_1 \vec{e}_1 + k_2 \vec{e}_2$, where e_1, e_2 are unit vectors in the direction of F_1 and F_2, resp.

$\underline{e_2 \cdot F} = e_2 (k_1 \vec{e}_1 + k_2 \vec{e}_2) = k_1 e_2 e_1 + k_2 e_2 e_2 =$

$0 + \underline{k_2} = \frac{10}{\sqrt{2}}$

$e_2 = \frac{1}{\sqrt{2}} \langle -1, -1 \rangle$

$k_2 = e_2 \cdot F = \frac{1}{\sqrt{2}} \langle -1, -1 \rangle \cdot \langle 0, -10 \rangle$

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