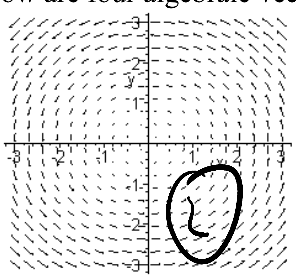
Math 2511: Calc III - Practice Exam 3

1. State the meaning or definitions of the following terms:
   a) vector field, conservative vector field, potential function of a vector field, volume, length of a curve, work, surface area, flux integral
   b) curl and divergence of a vector field F, gradient of a function
   c) $\iint_R dA$ or $\iint_R f(x,y)dA$ or $\iiint_Q f(x,y,z)dV$
   d) $\cancel{\iint_R dS}$ or $\int_C ds$ or $\int_C f(x,y)ds$ or $\int_C f(x,y)dx$ or $\int_C f(x,y)dy$ or $\cancel{\iint_S g(x,y,z)dS}$
   e) $\int_C \vec{F}\cdot d\vec{r}$ or $\cancel{\iint_S \vec{F}\cdot d\vec{S}}$
   f) $\int_C M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$
   g) What does it mean when a "line integral is independent of the path"?
   h) State the Fundamental Theorem of Line Integrals. Make sure to know when it applies, and when it helps.
   i) ~~State Green's Theorem. Make sure to know when it applies, and in what situation it helps.~~
   j) ~~State Gauss' Theorem. Make sure to know when it applies, and in what situation it helps.~~

2. Below are four algebraic vector fields and four sketches of vector fields. Match them.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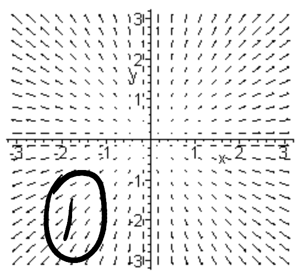
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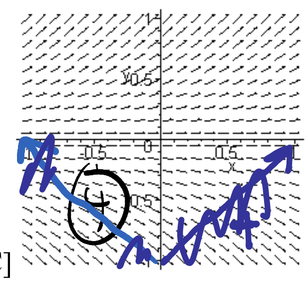
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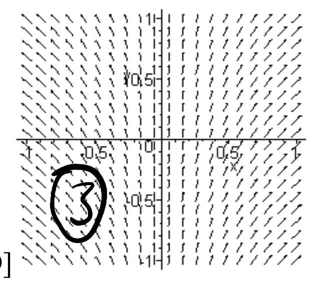
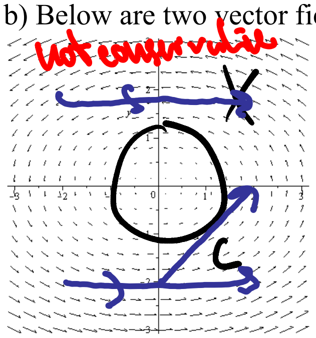
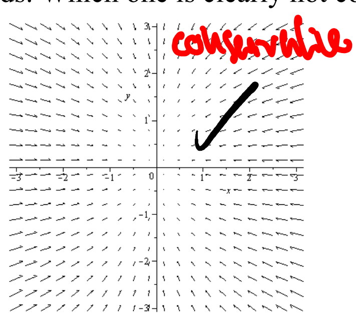

[D]

(1) $F(x,y) = \langle x, y \rangle$, (2) $F(x,y) = \langle -y, x \rangle$, (3) $F(x,y) = \langle x, 1 \rangle$, (4) $F(x,y) = \langle 1, y \rangle$

b) Below are two vector fields. Which one is clearly not conservative, and why?

$\oint F\,dr \neq 0$

c) Say in the vector field [C] above you integrate over a straight line from (0,-1) to ~~(-1,0)~~ (1,0) + is the integral positive, negative, or zero?
negative

from (-2,1) to (2,1) −
from (-2,-1) to (2,-1) +

3. Are the following statements true or false:
   a) If the divergence of a vector is zero, the vector field is conservative. F
   b) If $F(x,y,z)$ is a conservative vector field then $curl(F) = 0$ T
   c) If a line integral is independent of the path, then $\int_C F\cdot dr = 0$ for every path C F
   d) If a vector field is conservative then $\int_C F\cdot dr = 0$ for every closed path C T

e) $\iint_R dA$ denotes the surface area of the region R F (area)

f) ~~$\iint_R dS$ denotes the volume of the region R~~ ~~F (surface)~~

g) Can you apply the Fundamental Theorem of line integrals for the function $f(x,y,z) = xy\sin(z)\cos(x^2+y^2)$? F

h) Can you apply the Fundamental Theorem of line integrals for the vector field F
$F(x,y) = < 6xy^2 - 3x^2, 6x^2y + 3y^2 - 7 >$?

i) Can you apply Green's theorem for a curve C, which is a straight line from (0,0,0) to (1,2,3)? F

~~j) Can you apply the Divergence theorem to the plane $x+y+z=1$ over [-1, 1] x [-1, 1]?~~ F

4. Suppose that $F(x,y,z) = < x^3y^2z, x^2z, x^2y >$ is some vector field.
   a) Find div(F)
   $3x^2y^2z$

   b) Find curl(F)
   $(x^3y^2 - 2xy)\bar{e}_y + (2xz - 2x^3yz)\bar{e}_z$

   c) Find curl(curl(F))
   $-2x^3z\bar{e}_x + (-2z + 6x^2yz)\bar{e}_y + (3x^2y^2 - 2y)\bar{e}_z$

   d) Find div(curl(F))
   0

   e) grad., div., and curl of the vector field if appropriate for $< x^2, y^2, z^2 >$
   grad =n/a, div = 2x+2y+2z, curl = 0

   f) grad., div., and curl of the vector field if appropriate for $< \cos(y) + y\cos(x), \sin(x) - x\sin(y), xyz >$
   grad = n/a, div = $-y\sin(x) - x\cos(y) + xy$, curl = $(xz)\bar{e}_x - yz\bar{e}_y$

   g) grad., div., and curl of the vector field if appropriate for $f(x,y,z) = z\ln(x^2+y^2)$
   grad = $\dfrac{2zx}{x^2+y^2}\bar{e}_x + \dfrac{2zy}{x^2+y^2}\bar{e}_y + (\ln(x^2+y^2))\bar{e}_z$

5. Decide which of the following vector fields are conservative. If a vector is conservative, find its potential function
   a) $F(x,y) = < 2xy, x^2 >$
   ~~Not~~ conservative , $f(x,y) = x^2y + C$

   b) $F(x,y) = < e^x\cos(y), e^x\sin(y) >$
   Not conservative

   c) $F(x,y,z) = < \sin(y), -x\cos y, 1 >$
   Not conservative

   d) $F(x,y,z) = < 2xy, x^2+z^2, 2zy >$
   ~~Not~~ conservative

   $$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2+z^2 & 2zy \end{vmatrix} = <2z-2z, 0-0, 2x-2x>$$
   $f(x,y,z) = \underline{x^2y + z^2y + C}$

   e) $F(x,y) = < 6xy^2 - 3x^2, 6x^2y + 3y^2 - 7 >$

yes

f = 3x^2 y^2 – x^3 + y^3 -7x + C

f) $F(x,y) = < -2y^3 \sin(2x), 3y^2(1+\cos(2x)) >$ — $\frac{\partial N}{\partial x} = 3y^2(-2\sin(2x))$, $\frac{\partial M}{\partial y} = -6y^2\sin(2x)$

yes, conservative

g) $F(x,y) = < 4xy + z, 2x^2 + 6y, 2z >$

$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 4xy+z & 2x^2+6y & 2z \end{vmatrix} = <0-0, 4x+1$

Not conservative

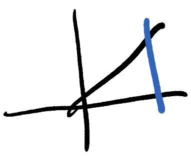
h) $F(x,y) = < 4xy + z^2, 2x^2 + 6yz, 2xz >$

Not conservative

$5) \int_0^1 \int_1^{y} x^2 y^3 \, dx\,dy = 25/84$

6. Evaluate the following integrals:

a) $\iint_R \cos(x^2)\, dA$ where R is the triangular region bounded by y = 0, y = x, and x = 1



$\int_0^1 \int_0^x \cos(x^2)\, dy\, dx = \frac{1}{2}\sin(1)$

b) $\iint_R dS$, where S is the portion of the hemisphere $f(x,y) = \sqrt{25 - x^2 - y^2}$ that lies above the circle $x^2 + y^2 \le 9$

$\iint_C \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint \frac{5}{\sqrt{25-x^2-y^2}}\, dA = \int_0^{2\pi}\int_0^{3} \frac{5}{\sqrt{25-r^2}}\, r\, dr\, d\theta = 10\pi$

c) $\int_C x^2 - y + 3z\, ds$ where C is a line segment given by $r(t) = < t, 2t, 3t >$, $0 \le t \le 1$

$\int_0^1 (t^2 - 2t + 9t)\sqrt{1+4+9}\, dt = \sqrt{14} \cdot 23/6$

d) $\int_C F \cdot dr$ where $F(x,y) = < y, x^2 >$ and C is the curve given by $r(t) = < 4-t, 4t - t^2 >$, $0 \le t \le 3$

$\int_C y\,dx + x^2\,dy = \int_0^3 (4t - t^2)(-1)\,dt + (4-t)^2(4-2t)\,dt = \frac{15}{2}$

$\int_0^3 (-4t + t^2 + (4-t)^2(4-2t))\, dt = \frac{15}{2}$

see at the end

e) $\int_C y\,dx + x^2\,dy$ where C is a parabolic arc given by $r(t) = < t, 1 - t^2 >$, $-1 \le t \le 1$

$\int_{-1}^{1} (1-t^2)\,dt + t^2(-2t)\,dt = 4/3$

f) $\iint_S (x+z)\,dS$ where S is the first-octant portion of the cylinder $y^2 + z^2 = 9$ between x = 0 and x = 4

can't do, need surface as z = f(x,y)

g) Find the flux of the vector field $F(x,y,z) = < x, y, z >$ through the surface given by potion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane. Note that this surface is *not* closed.

$$\cancel{\iint_R -f_x M - f_y N + P\, dA = \iint_{\text{circle}} 2x^2+2y^2+4-x^2-y^2\, dA = \int_0^{2\pi}\int_0^2 (4+r^2)\, r\, dr\, d\theta = 24\pi}$$

7. For the following line integrals there is a short-cut you can use to simplify your computations (but justify your shortcut by quoting the appropriate theorem)

   a) $\int_C F \cdot dr$ where $F(x,y,z) = < e^x \cos(y), -e^x \sin(y) >$ and C is the curve $r(t) = < 2\cos(t), 2\sin(t) >$, $0 \le t \le 2\pi$

   F conservative, C closed curve $\Rightarrow \int \vec{F}\, d\vec{r} = 0$

   b) $\int_C 2xyz\,dx + x^2 z\,dy + x^2 y\,dz$ where C is some smooth curve from (0,0,0) to (1,4,3)

   $= f(1,4,3) - f(0,0,0) = 12$, where $f(x,y,z) = x^2 y z$ is potential function

   c) $\int_C F \cdot dr$ where $F(x,y) = < y^3 + 1, 3xy^2 + 1 >$ and C is the upper half of the unit circle, from (1,0) to (-1,0).

   $= f(-1,0) - f(1,0) = -2$, where $f(x,y) = xy^3 + x$ is potential function

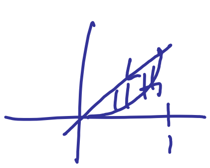
   or integrate along $r(t) = \langle t, 0\rangle$, $t = -1 \ldots 1$: $\int_{+1}^{-1} 0+1\, dt + (0+1)\cdot 0\, dt = \underline{\underline{-2}}$

   d) $\int_C F \cdot dr$ where $F(x,y) = < y^3 x, 3xy^2 >$ and C is the line segment from (-1,0) to (2,3).

   $r(t) = \langle -1, 0\rangle + t\langle 3,3\rangle = \langle -1+3t, 3t\rangle$ (x, y)

   $\int_0^1 (3t)^3(-1+3t)\cdot 3\,dt + 3(-1+3t)(3t)^2 \cdot 3\,dt = \underline{\underline{\frac{621}{10}}}$

   e) $\int_C y^3 dx + (x^3 + 3xy^2) dy$ where C is the path from (0,0) to (1,1) along the graph of $y = x^3$ and from (1,1) to (0,0) along the graph of $y = x$.



   Green: $\iint_R 3x^2 + 3y^2 - 3y^2\, dA = 3\int_0^1\int_{x^3}^{x} x^2\, dy\, dx = \frac{1}{4}$

   f) $\iint_S \vec{F} \cdot \vec{n}\, dS$ where $F(x,y,z) = < x, y, z >$ and S is $x^2 + y^2 + z^2 = 4$

   $$\cancel{\text{Gauss: } \iint \vec{F}\cdot\vec{n}\, dS = \iiint \operatorname{div}(\vec{F})\, dV = \iiint 3\, dV = 3\cdot\frac{4}{3}\pi(2)^3}$$

8. Green's Theorem

a) Use Green's theorem to find $\int_C F \cdot dr$ where $F(x,y) = < y^3, x^3 + 3xy^2 >$ and C is the circle with radius 3, oriented counter-clockwise (You may need the double-angle formula for cos somewhere during your computations)

$$\iint_R 3x^2 + 3y^2 - 3y^2 \, dA = \int_0^{2\pi}\int_0^3 3r^2\cos^2\theta \, r\,dr\,d\theta = \frac{243}{4}\pi$$

b) Evaluate $\iint_R dA$ where R is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ by using a vector field $F(x,y) = < -\frac{y}{2}, \frac{x}{2} >$ and the boundary C of the ellipse R.

$r(t) = \langle 2\cos(t), 3\sin(t) \rangle, \ t \in [0, 2\pi]$

$$\iint \frac{1}{2} - \left(-\frac{1}{2}\right) dA = \int \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle d\vec{r} = \frac{1}{2}\int -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} -3\sin(t)\cdot(-2\sin(t)) + 2\cos(t)\cdot 3\cos(t)\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} 6 \, dt = 6\pi$$

9. Evaluate the following integrals. You can use any theorem that's appropriate:

c) $\int_C 2xyzdx + x^2zdy + x^2ydz$ where C is a smooth curve from (0,0,0) to (1,4,3)

Already done above

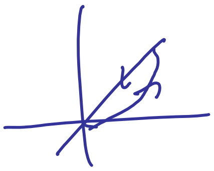
d) $\int_C ydx + 2xdy$ where C is the boundary of the square with vertices (0,0), (0,2), (2,0), and (2,2)

Green's $\iint_R 2 - 1 \, dA = \text{area (square)} = 4$

e) $\int_C xy^2dx + x^2ydy$, where C is given by $r(t) = < 4\cos(t), 2\sin(t) >$, t between 0 and 2 Pi.

Green $\iint_R 2xy - 2xy \, dA = 0$ (also: closed curve + conservative vector field

f) $\int_C xydx + x^2dy$ where C is the boundary of the region between the graphs of $y = x^2$ and $y = x$.



Green $\iint 2x - x \, dA = \int_0^1 \int_{x^2}^{x} x \, dy\,dx = \frac{1}{12}$

10. Prove the following:

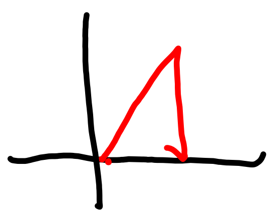
a) If $F(x, y, z) = < M(x, y, z), N(x, y, z), P(x, y, z) >$ is any vector field where M, N, P are twice continuously differentiable then $div(curl(F)) = 0$

just work out all these partials

b) A function (not a vector field) $f(x, y, z)$ is called harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$. Show that for any function $f(x, y, z)$ the function $\frac{1}{f(x, y, z)}$ is harmonic.

#6

a) $\iint_R \cos(x^2)\, dA$ R:



$= \frac{1}{2}$

$= \int_0^1 \int_0^x \cos(x^2)\, dy\, dx = \int_0^1 x\cos(x^2)\, dx = \frac{1}{2}\cos(x^2)\Big|_0^1$

b) $\int_0^1 \int_1^{2y} x^2 y^3 \, dx\, dy = \frac{25}{84}$

c) $\int_C ds = \int_0^2 \sqrt{4t^2+1}\, dt = \sqrt{17} - \frac{1}{4}\ln(\sqrt{17} - 4)$ with Maple

d) $\int_C x^2 y^3 \, dx = \int_0^2 (t^2)^2 (t^3)^3 \, 2t \, dt = 2\int_0^2 t^{14} \, dt = \underline{\frac{2}{15} \cdot 2^{15}}$

e) $\int_C x^2 - y + 3 \, ds = \int_0^{\pi} \left(4\cos^2(t) - 2\sin(t) + 3\right) \sqrt{4\cos^2 + 4\sin^2} \, dt$

$= 4 \int_0^{\pi} 4\cos^2(t) - 2\sin(t) + 3 \, dt = \underline{-4 + 5\pi}$

f) $\int_C x^2 - y + 3z \, ds = \int_0^1 \left(t^2 - 2t + 9t \right) \sqrt{1+4+9} = \sqrt{14} \, \frac{23}{6}$

g) $\int_C F \, dr = \int y \, dx + x^2 \, dy = \int_0^1 (4t - t^2)(-1) \, dt + (4-t)^2 (4-2t) \, dt$

$= 73/2$

h) $\int_C F \, dr = \int yz \, dx + x^2 \, dy + zy \, dz =$

$= \int_1^3 3t(2-t^2)(-1) + (1-t)^2 \cdot 3 + (2-t^2)(3t)(-2t) \, dt$

$= \frac{249}{20}$

i) $\int_C y \, dx + x^3 \, dy = \int_{-1}^1 (1-t^2) \, 1 + t^3(-2t) \, dt = 8/15$