Panel 1

Last Time

Function $f(x,y)$ — $\int_C f(x,y)\,dx$, $\int_C f(x,y)\,dy$, $\int_C f(x,y)\,ds = \int f(x(t),y(t))\sqrt{(x')^2+(y')^2}\,dt$ (*)

Vector field $\vec{F} = \langle M,N\rangle$ — work $\Rightarrow$ $\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy$

Fundamental Thm. of Line Integration:

$\int_C \vec{F}\,d\vec{r} = f(\gamma(b)) - f(\gamma(a))$, f is potential of F

If $\vec{F}$ is conservative!

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Panel 2

Fundamental Theorem for Line Integrals

If F is conservative with potential function f, and $\gamma(t)$, $a \le t \le b$, a smooth curve. Then:

$$\int_\gamma \vec{F}\,d\vec{r} = f(\gamma(b)) - f(\gamma(a))$$

<u>Ex</u>: $\int_C x^2\,dx + y^2\,dy$, C from $(0,1)$ to $(5,4)$ $= \frac{1}{3}(5^3+4^3) - \frac{1}{3}(0^3+1^3)$

Conservative? $\frac{\partial N}{\partial x} \overset{?}{=} \frac{\partial M}{\partial x}$

$0 = 0$ ✓

$F = \langle x^2, y^2\rangle$

$f_x = x^2 \Rightarrow f = \frac{1}{3}x^3 + C(y)$

$f_y = C'(y) = y^2 \Rightarrow C(y) = \frac{1}{3}y^3$

$f(x,y) = \frac{1}{3}(x^3+y^3)$

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Panel 3

Conservative Vector Fields

$$\int_{\gamma_1} \vec{F}\, d\vec{r} = \int_{\gamma_2} \vec{F}\, d\vec{r}$$

$$\cancel{\int_{\gamma_2} \vec{F}\, d\vec{r}}$$

Nice! $\oint_{\gamma} \vec{F}\, d\vec{r} = 0$

Thm: A vector field $\vec{F} = \langle M, N \rangle$ is conservative, iff $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ or $\text{curl}(\vec{F}) = 0$

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Panel 4

Ex: Find $\oint_{\gamma} \tan(y)\, dx + x \sec^2(y)\, dy = 0$

where $\gamma(t) = \langle \underbrace{\cos(t)}_{x}, \underbrace{\sin(t)}_{y} \rangle$, $t \in [0, 2\pi]$

① $\int_0^{2\pi} \tan(\sin(t))(-\sin(t))dt + \cos(t) \sec^2(\sin(t)) \cdot \cos(t) dt$

② Realize $\gamma(t)$ is circle. AHA!!! Is $\vec{F}$ conservative !?!

$\frac{\partial N}{\partial x} = \sec^2(y)$ $\frac{\partial M}{\partial y} = \sec^2(y)$

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Panel 5

$\int \vec{F}\, d\vec{r}$ important $\Rightarrow$ Work

$\int_C \vec{F}\, d\vec{r}$ — long way / Fund. thm (if conservative)

X $\oint_C \vec{F}\, d\vec{r}$ — long way / zero (if conservative) / else ...

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Panel 6

$\vec{F} = \langle t, y, -x \rangle$, $\vec{r}(t) = \langle t, \sin(t), \cos(t) \rangle$, $t \in [0, \pi]$

$\int_C \vec{F}\, d\vec{r} =$ (not closed)

$\int_C t\, dx + y\, dy - x\, dz = \int_0^\pi \cos(t)\,dt + \sin(t)\cos(t)\,dt + t\sin(t)\,dt$

int(cos(t) + sin(t)·cos(t) + t·sin(t), t = 0..Pi)

$-2\sin + \pi$

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Panel 7

$\int_C (yz)\,dx + (xz)\,dy + (xy+2z)\,dz$, where C is segment from $(1,0,-2)$ to $(4,6,3)$

$= xyz + z^2 \Big|_{(1,0,-2)}^{(4,6,3)} = 4\cdot6\cdot3+9-(0+4)$

① $\vec{r}(t) = (1,0,-2) + t\left((4,6,3)-(1,0,-2)\right) = (1,0,-2)+t(3,6,5)$

$= (\underbrace{1+3t}_{x}, \underbrace{6t}_{y}, \underbrace{-2+5t}_{z}),\quad t\in[0,1]$

$\int_0^1 (6t)(-2+5t)3\,dt + (1+3t)(-2+5t)6\,dt + \left((1+3t)(6t) + 2(-2+5t)\right)5\,dt$

② F cons?

~~curl(F)=0?~~

$f_x = yz \Rightarrow f = xyz + C(y,z)$

$f_y = \cancel{xz} + C_y(y,z) = \cancel{xz} \Rightarrow C_y(y,z)=0 \Rightarrow C = C(z)$

$f_z = \cancel{xy} + C'(z) = \cancel{xy} + 2z \Rightarrow C(z) = z^2 + c$

$\boxed{f = xyz + z^2 + c}$

φ: label!

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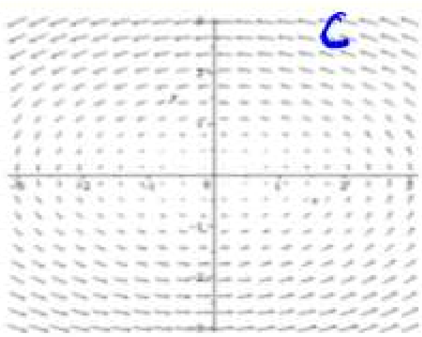
Panel 8

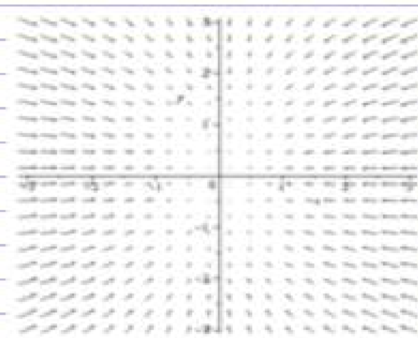
Last Quiz

Name: ______

① One of these vector fields is conservative. Which one?

C





② Say in the vector field [C] above you integrate over a straight line from (0,-1) to (-1,0). Is the integral positive, negative, or zero?

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Panel 9

② Evaluate these integrals:

a) $\int_C xy\, ds$ where $C: \gamma(t) = \langle 1+2t, 3t \rangle$, $t \in [0,1]$

b) $\int_C y\, dx + x\, dy$, C from $(-1,-1)$ to $(1,1)$

c) $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle 4xy - 3y^2, 2x^2 - 6xy \rangle$ and C is the rectangle $(2,2), (-2,2), (-2,-2), (2,-2)$

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Panel 10

Green's Theorem: R a region in xy-plane with boundary curve C. C is piecewise smooth, non-intersecting, closed, and positively oriented. $\vec{F} = \langle M, N \rangle$ is a smooth vector field. Then:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Note: If F conservative $\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

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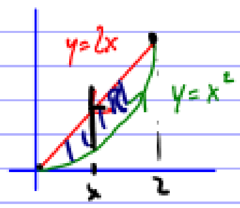
Panel 11

Ex: Evaluate $\oint_C 5xy\,dx + x^3\,dy$, where C is as shown:



Method A: Green's theorem

$$\oint_C 5xy\,dx + x^3\,dy =$$

$\boxed{y=x^2 \quad r(t)=\langle t,t^2\rangle,}$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R 3x^2 - 5x\,dA =$$

$$= \int_0^2\int_{x^2}^{2x} 3x^2 - 5x\,dy\,dx = \ldots - \frac{28}{15}$$

$\boxed{y=2x \quad r(t)=\langle t,2t\rangle}$

Method B: $\displaystyle\int_{\{y=x^2\}} 5xy\,dx + x^3\,dy + \int_{\{y=2x\}} 5xy\,dx + x^3\,dy =$

$$\int_0^2 5\cdot t\cdot t^2\,dt + t^3\,2t\,dt + \int_2^0 5t\,2t\,dt + t^3\,2\,dt$$

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Panel 12

Ex: Evaluate $\oint_C \overset{M}{2xy}\,dx + \overset{N}{(x^2+y^2)}\,dy$, C is $4x^2+9y^2=36$

closed curve ⇒ Green!

$$= \iint_{?} 2x - 2x\,dA = 0$$

Because F is conservative, and $\oint F\,dr = 0$ ✓

Green checks this automatically!

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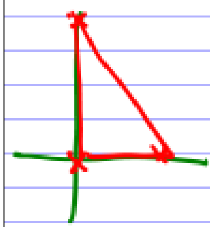
Panel 13

Ex: Find $\oint_\gamma (x\sin(y^2)-y)\,dx + (\underbrace{x^2 y\cos(y^2)+3x})$

where γ is the triangle $(0,0)$, $(1,0)$, $(0,1)$.



$$\iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA =$$

$$\iint \left[\cancel{2xy\cos(y^2)}+3\right] - \left[\cancel{2yx\cos(y^2)}-1\right] dA =$$

$$\iint_\Delta 4\,dA = 4\iint_\Delta dA = 4\,\text{area}(\Delta) = 4\cdot\frac{1}{2} = \underline{2}$$

Note: $\iint_R dA = \text{area}(R)$.

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Panel 14

Evaluate $\oint_C (\overbrace{3y - e^{\sin(x)}}^{M})\,dx + (\overbrace{7x+\sqrt{y^4+1}}^{N})\,dy$

where C is the circle $x^2+y^2=9$

$$\iint_{\text{circle}} 7-3\,dA = 4\iint_{\text{circle}} dA = 4\cdot\pi 3^2 = 4\cdot 9\pi = \underline{36\pi}$$

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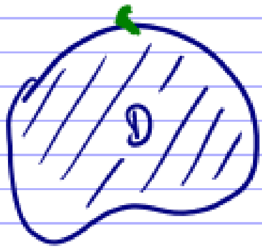
Panel 15

Theorem: If D is a region enclosed by a curve C then $\text{area}(D) = \frac{1}{2}\oint_C x\,dy - y\,dx$



Proof: $\oint_C \overset{N}{x}\,dy \overset{M}{-y}\,dx =$

$$\iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA = \iint 1-(-1)\,dA =$$

$$= 2\iint_D dA = 2\,\text{area}(D)$$

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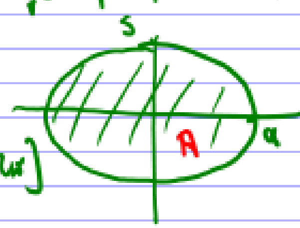
Panel 16

Ex: Find area enclosed by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Ellipse!

$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \Rightarrow x^2+y^2=r^2$

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$



$r(t) = \langle \underset{x}{a\cos(t)}, \underset{y}{b\sin(t)} \rangle, \ t\in[0,2\pi]$

$$A = \frac{1}{2}\int_0^{2\pi} a\cos(t)\cdot b\cos(t)\,dt + b\sin(t)\cdot a\sin(t)\,dt =$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,(\cos^2(t)+\sin^2(t))\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{1}{2}ab\cdot 2\pi =$$

$$= \pi ab$$

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Panel 17

Ex: Evaluate $\oint_C y^2\,dx + 3xy\,dy$ where C is the boundary of the region between $x^2+y^2=1$ and $x^2+y^2=4$.

HW

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Panel 18

Integration Summary

Line Integration

function: $\int f\,dx$, $\int f\,dy$, $\int f\,ds$

vector field: $\int \vec{F}\cdot d\vec{r}$

$\int M\,dx + N\,dy\ (+P\,dz)$

conserv.: $f(b)-f(a)$, f potential

closed curve: $\iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$ (Green)

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