Panel 1

Last Time:

Vector fields: $\vec{F}:\mathbb{R}^2\to\mathbb{R}^2$, $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$

$$\mathrm{div}(\vec{F}) = \nabla\cdot\vec{F} = \langle \partial_x, \partial_y\rangle \cdot \langle M, N\rangle = \partial_x M + \partial_y N$$

$$\mathrm{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle \quad , -(\quad), \quad \rangle$$

anti-deriv.

Conservative vector field: $\vec{F}$ is conservative if $\vec{F} = \nabla f$.

$\vec{F}$ cons. $\Rightarrow \mathrm{curl}(\vec{F}) = 0$

in $\mathbb{R}^2$: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

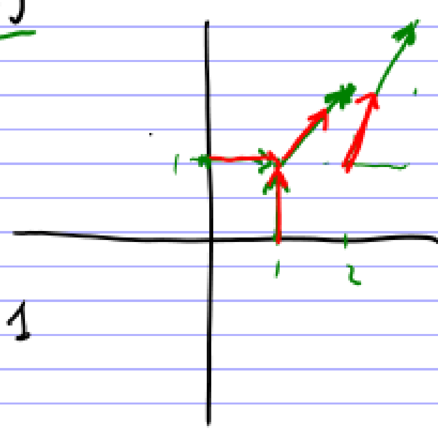
1

Panel 2

$$\vec{F}(x,y) = \frac{1}{(x^2+y^2)^{1/2}} \langle y, x\rangle$$

1, 0

0, 2



$$\|\vec{F}\| = \frac{1}{(x^2+y^2)^{1/2}} \|\langle y, x\rangle\| = 1$$

2

Panel 3

$\vec{F} = \langle 3x^2 + 2y^2, 4xy + 3 \rangle$ is conservative.

$$\frac{\partial M}{\partial y} = 4y = \frac{\partial N}{\partial x} = 4y$$. There is a function f

$f_x = M \Rightarrow f_x = 3x^2 + 2y^2 \Rightarrow f = \int 3x^2 + 2y^2 \, dx = x^3 + 2y^2 x + C(y)$

$\underline{f_y = N}$

$$f_y = \cancel{4y}x + C'(y) \overset{\text{want}}{=} 4\cancel{xy} + 3$$

$$\Rightarrow C'(y) = 3 \Rightarrow C(y) = 3y + c$$

$\underline{f(x,y) = x^3 + 2y^2 x + 3y + c}$ is potential

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Panel 4

$$\vec{F} = \langle \overset{M}{e^x}, \overset{N}{e^{xy}}, \overset{P}{e^{xyz}} \rangle$$

$$\text{div}(\vec{F}) = M_x + N_y + P_z = e^x + xe^{xy} + xye^{xyz}$$

$$\text{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ e^x & e^{xy} & e^{xyz} \end{vmatrix} = \langle xze^{xyz} - 0, -(yze^{xyz} - 0), ye^{xy} \rangle$$

$$= \underline{\langle xze^{xyz}, -yze^{xyz}, ye^{xy} \rangle}$$

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Panel 5

(1) If $f: \mathbb{R}^3 \to \mathbb{R}$ is a function and $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, which expression is meaningful:

$f(x,y,z) = x^2 y + z^2 x$

$\vec{F} = \langle x, yx, x^2 - z^2 \rangle$

- $\cancel{\text{curl}(f)}$
- grad(f)
- div(F)
- curl(grad(f))
- $\cancel{\text{grad}(F)}$
- grad(div(F))
- div(grad(f))
- $\cancel{\text{grad}(\text{div}(f))}$
- curl(curl(F))
- div(div(F))
- (grad(f)) $\times$ (div(F))
- div(curl(grad(f)))

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Panel 6

Ex: Take $f(x,y,z) = x^2 y^2 - xz$ and

$\vec{F}(x,y,z) = \langle x^2, xz, yz^2 \rangle$

$\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$

Find the quantities that make sense:

$\nabla f =$

$\nabla \cdot F$ = div

$\nabla \times F$ = curl

$\cancel{\nabla \vec{F}}$

$\cancel{\nabla \cdot f}$

$\cancel{F \times f}$

$f F$

$2\langle x, y \rangle$

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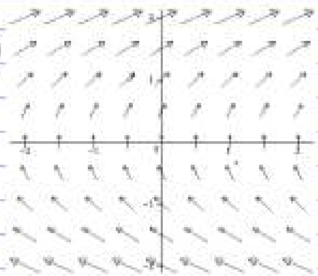
Panel 7

Name: ________

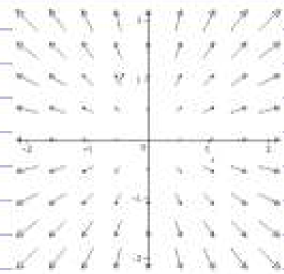
Quiz

① Match the algebraic expressions with the fieldplots:

(i)



(ii)



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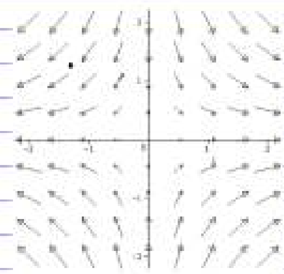

A) $F(x,y) = \langle x, -y \rangle$

C) $F(x,y) = \langle y, 1 \rangle$

B) $F(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2+4}}, \frac{y}{\sqrt{x^2+y^2+4}} \right\rangle$

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Panel 8

② Suppose $F(x,y,z) = \langle z^2 - x^2, 2xy, xy+3 \rangle$. Find

a) $\mathrm{div}(F)$

b) $\mathrm{curl}(F)$

③ Find potential function for $\vec{F} = (3 + 2xy, x^2 - 3y^2)$ if it exists.

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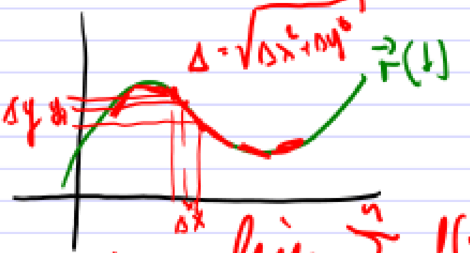
Panel 9

Line Integrals

Suppose $r(t) = \langle g(t), h(t) \rangle$ describes a curve C in $\mathbb{R}^2$ and $f(x,y)$ is a function defined on C. Then we define the line integral of f along C as:

$$\int_C f(x,y)\, ds$$

Def



$$\sqrt{\Delta x^2 + \Delta y^2} =$$

$$\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t$$

$$\int f(x,y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t$$

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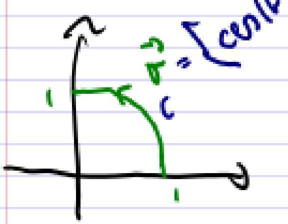
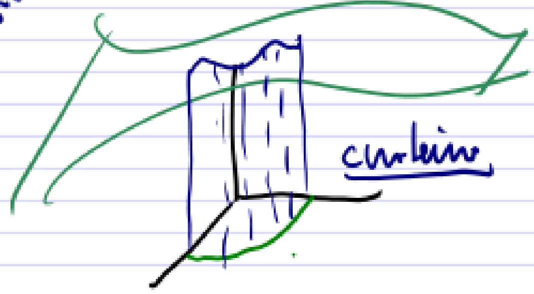
Panel 10

Ex: Find $\int x y^2\, ds$, where $\vec{r}$ describes first quarter circle, radius 1.





$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt$$

$\vec{r}(t) = \langle x(t), y(t) \rangle = \langle \cos(t), \sin(t) \rangle$

$$= \int_0^{\pi/2} \cos(t) \cdot \sin^2(t) \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt$$

$$= \int_0 \cos(t) \sin^2(t)\, dt = \ ?$$

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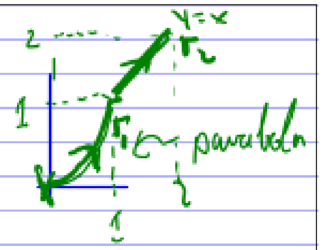
Panel 11



Ex: Evaluate $\int_C 2x\,ds$ where C:

$\int 2x\,ds =$ $\qquad ds = \sqrt{(x')^2+(y')^2}\,dt$

$\int_{r_1} 2x\,ds + \int_{r_2} 2x\,ds$ $\qquad r_1(t) = \langle t, t^2\rangle$, $t\in[0,1]$

$\int_0^1 2t\sqrt{1+4t^2}\,dt +$ $\qquad r_2(t) = \langle t, t\rangle$, $t\in[1,2]$

$\int_1^2 2t\sqrt{1+1}\,dt$

$\boxed{12.10}$

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Panel 12

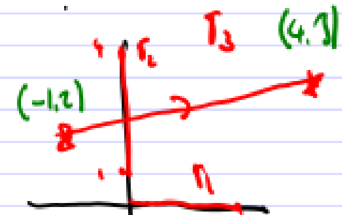
Question: What is $\int_C f(x,y)\,ds = \int_C xy\,ds$

a) C goes along x-axis from 0 to 2

b) C goes along y-axis from 1 to 4

c) C line from (-1,2) to (4,3)

(1) Find curve in $\vec{r}(t) = \langle ?, ?\rangle$ form.



a) $r(t) = \langle t, 0\rangle$ $t\in[0,2]$

b) $r(t) = \langle 0, t\rangle$, $t\in[1,4]$

c) $r(t) = (-1,2) + t\big((4,3)-(-1,2)\big) = (-1,2) + t(5,1)$, $t\in[0,1]$

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Panel 13

Side Note: To parametrize a curve, you can usually get by with

$r(t) = \langle t, f(t) \rangle$ (if $y = f(x)$ describes curve)

$r(t) = \langle r\cos(t), r\sin(t) \rangle$, circle radius r

line from $A = (a_1, a_2)$ to $B = (b_1, b_2)$

$r(t) = A + t(B - A)$, $t \in [0,1]$

$y = mx + b$

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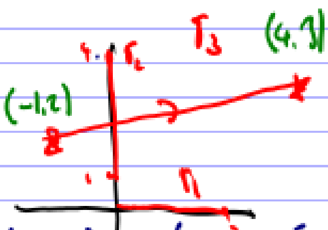
Panel 14

$\int xy \, ds = \int x(t)\, y(t) \sqrt{(x')^2 + (y')^2}\, dt$

a) $r(t) = \langle t, 0 \rangle$ $t \in [0,2]$

b) $r(t) = \langle 0, t \rangle$, $t \in [1,4]$



c) $r(t) = (-1,2) + t((4,3) - (-1,2)) = (-1,2) + t(5,1)$, $t \in [0,1]$

$= (-1 + 5t, 2 + t)$

$\Rightarrow$ a) $\int_0^2 t \cdot 0 \sqrt{1}\, dt = 0$

c) $\int_0^1 (-1+5t)(2+t)\sqrt{25+1}\, dt$

b) $\int 0 \cdot t \sqrt{1}\, dt = 0$

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Panel 15

We also define two variations of line integrals:

Let C be the curve given by $\vec{r}(t) = \langle x(t), y(t) \rangle$.

Def:

$$\int_C f(x,y)\, ds = \int f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt$$

$$\int_C f(x,y)\, dx = \int f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x,y)\, dy = \int f(x(t), y(t))\, y'(t)\, dt$$

$\parallel$ not $\int_{C} f(x,y)\, dx$

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Panel 16

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Panel 17

Ex: Find $\int_C xy^2\,dx$ and $\int_C xy\,dy$ where C is parabola from $(0,0)$ to $(2,4)$ $\Rightarrow y = f(x) = x^2$

x y

$r(t) = \langle t, t^2 \rangle$

$$\int_C xy^2\,dx = \int_0^2 t\,(t^2)^2\, 1 \cdot dt$$

$$\int_C xy^2\,dy = \int_0^2 t\,(t^2)^2\, 2t\,dt$$

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