Panel 1

$$\int_a^b f(x)\,dx = 0 \Rightarrow \text{area under } f(x) \text{ is zero} \Rightarrow f = 0$$

$$\int_{-a}^{a} x^3\,dx = \frac{1}{4}x^4 \Big|_{-a}^{a} = 0$$

1

Panel 2

Birds-Eye View so far

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2 - 7x + 2$

$f: \mathbb{R} \to \mathbb{R}^{2/3}$ $r(t) = \langle t, t^2 \rangle, t^3\rangle$ (space) curve

$f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x^2y + y + 7$

$\mathbb{R}^3 \to \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$ $f(x,y) = \langle xy, x^2+y^2 \rangle$

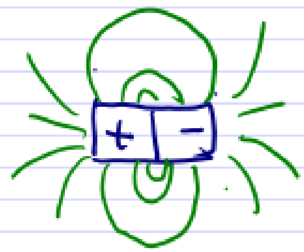
2

Panel 3

Vector Fields: If for each point P in a region R there is a unique vector having initial point P, then the totality of such vectors is called a **vector field**

Ex: A river.

Ex: Magnetic Field



Ex: Gravity

3

Panel 4

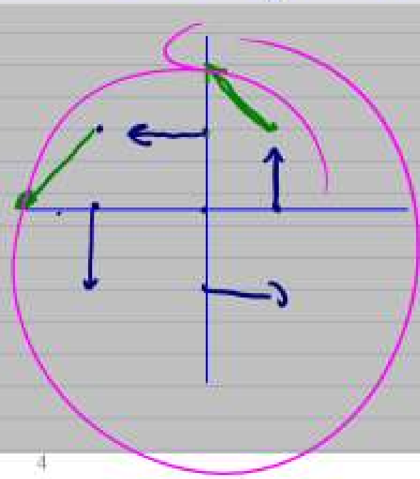
Mathematically, a vector field is given as:

$$\vec{F}(x,y) = \langle N(x,y), M(x,y) \rangle = N\vec{i} + M\vec{j}$$

$$\vec{F}(x,y,z) = \langle N(x,y,z), N(x,y,z), P(x,y,z) \rangle = N\vec{i} + M\vec{j} + P\vec{k}$$

Ex: Describe $\vec{F}(x,y) = \langle -y, x \rangle = -y\vec{i} + x\vec{j}$

(x,y)	$\vec{F}(x,y)$
$(0,0)$	$(0,0)$
$(1,0)$	$\langle 0,1 \rangle$
$(0,1)$	$\langle -1,0 \rangle$
$(1,1)$	$\langle -1,1 \rangle$
$(-1,1)$	$\langle -1,-1 \rangle$



4

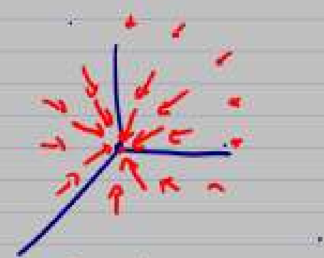
Panel 5

Def: If $r(x,y,z) = \langle x,y,z \rangle$ then $F(x,y,z) = \frac{c}{\|r\|^2} \vec{u}$ where $u = r/\|r\|$ is called inverse square field.

Ex: Describe inverse square field for $c = -1$.

$$\vec{F}(x,y,z) = \frac{-1}{(x^2+y^2+z^2)} \frac{\langle x,y,z \rangle}{(x^2+y^2+z^2)^{1/2}} = \frac{-\langle x,y,z \rangle}{(x^2+y^2+z^2)^{3/2}}$$

I am at (x,y,z). Vector field pulls me back to origin.
It is stronger the closer I am to zero.



Gravity

5

Panel 6

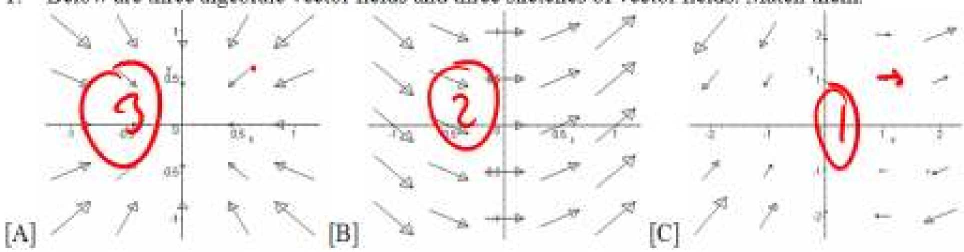
Maple offers "fieldplot" and "fieldplot3d"

```
> with(plots);
> fieldplot([-y, x], x = -2..2, y = -2..2);
> fieldplot3d([-x/(x^2+y^2+z^2)^(3/2), -y/(x^2+y^2+z^2)^(3/2), -z/(x^2+y^2+z^2)^(3/2)], x = -2..2, y = -2..2, z = -2..2);
>
```

6

Panel 7

1. Below are three algebraic vector fields and three sketches of vector fields. Match them.



(1) $F(x,y) = \langle xy, y(x-1) \rangle$

(2) $F(x,y) = \langle 1, x \rangle$

(3) $F(x,y) = \langle -x, -y \rangle$

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Panel 8

Def: Suppose $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$

Then

$$\operatorname{curl}(\vec{F}) = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

curl of F

$$\operatorname{div}(\vec{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

divergence

$\operatorname{grad}(\operatorname{div}(\vec{F}))$ or $\operatorname{div}(\operatorname{grad}(f))$

vector field

function

8

Panel 9

Ex: Let $F(x,y,z) = \langle xy,\ yz,\ xz \rangle$. Then

curl(F):

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z},\ \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right),\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(yz),\ (z-0),\ 0-x \right\rangle$$

$$\underline{\langle -y, -z, -x \rangle}$$

$$\text{div}(F) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + \underline{z + x}$$

Panel 10

Ex: $F(x,y,z) = \langle xy^2z^4,\ 2x^2y + z,\ y^3z^2 \rangle$

Find curl(F) and div(F)

HW

Note: curl$(\vec{F}) = \nabla \times F$ $\qquad M_x + N_y + P_z$

div$(\vec{F}) = \nabla \cdot F = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle M, N, P \rangle$

grad$(f) = \nabla = \langle \partial_x, \partial_y, \partial_z \rangle$

Panel 11

2. Suppose that $F(x,y,z) = \langle x^3z, x^2z, xy \rangle$ is some vector field.

   a) Find $div(F)$

$$\nabla \cdot \vec{F} = \underline{3x^2z + 0} + 0$$

   b) Find $curl(F)$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^3z & x^2z & xy \end{vmatrix} = \langle x - x^2, -(y - x^3), 2xz - 0 \rangle$$

$$= \underline{\langle x - x^2, x^3 - y, 2xz \rangle}$$

11

Panel 12

Def: A vector field $\vec{F}$ is conservative if there is a function $f : \mathbb{R}^3 \to \mathbb{R}$ s.t. $\nabla f = \vec{F}$.

The function f is the potential (antiderivative) function for $\vec{F}$

Ex: Find vector field with potential

$f(x,y,z) = x^2 - 3y^2 + 4z^2$

$$\vec{F} = \nabla f = \nabla(x^2 - 3y^2 + 4z^2) = \langle 2x, -6y, 8z \rangle$$

$x^2 - 3y^2 + 4z^2$ is potential for $\langle 2x, -6y, 8z \rangle$

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Panel 13

Which of the following vector field(s) has as potential function $f(x,y,z) = x^2y^2z^2 + xy + zy$

(a) $\vec{F} = \langle 2x, \cancel{y}, z \rangle$

(b) $\vec{F} = \langle 2xy\cancel{^2z^2} + x + y \rangle$

$\cancel{(c)}$ $\vec{F} = \langle 2xy^2z^2 + y, 2x^2yz^2 + x + z, 2x^2y^2z + y \rangle$

$\cancel{(d)}$ $\vec{F} = \langle 2xy^2z^2, x, y \rangle$

13

Panel 14

Suppose $\vec{F}$ is conservative, i.e. $\vec{F} = \langle f_x, f_y, f_z \rangle$. Then

$$\text{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, (f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle = \langle 0, 0, 0 \rangle$$

Theorem: If a vector field $\vec{F}$ wants to be conservative, $\text{curl}(\vec{F}) = 0$

If $\vec{F}$ is 2D vector field, then

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}, \text{ where } \vec{F} = \langle M, N, 0 \rangle$$

$$\begin{matrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M(x,y) & N(x,y) & 0 \end{matrix}$$

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Panel 15

Which of the following vector fields is ~~not~~ conservative

(a) $F(x,y) = \langle x, y \rangle$ ✓

(b) $F(x,y) = \langle x^2+y^2, 2xy \rangle$ ✓

∂_x $-e^x\sin(y) \overset{?}{=} -e^x\sin(y)$

(c) $F(x,y) = \langle \underline{e^x\cos(y)}, \underline{-e^x\sin(y)} \rangle$ ✓

~~(d)~~ $F(x,y) = \langle x^2\cos(y), -y^2\sin(x) \rangle$ Not conservative

$-x^2\sin(y) \neq -y^2\cos(x)$

15

Panel 16

Find potential function for $\vec{F} = \langle \overset{M}{3+2xy}, \overset{N}{\underline{x^2-3y^2}} \rangle$ if exists

① Check $N_x = M_y$: $2x \overset{?}{=} 2x$ ✓

Want to find $f(x,y)$ s.t.

$f_x = 3+2xy \Rightarrow f(x,y) = \int 3+2xy\,dx = 3x+x^2y+C(y)$

$\Rightarrow f(x,y) = \underline{3x} + \underline{x^2y} + \underline{C(y)}$

$f_y = \cancel{x^2} + C'(y) \overset{!}{=} \cancel{x^2} - 3y^2$

$\Rightarrow C'(y) = -3y^2 \Rightarrow C(y) = -y^3 + c$

Potential: $f(x,y) = 3x + x^2y - y^3 + c$

16 12:0

Panel 17

Find potential function for $\langle \overset{M}{x^2\cos(y)}, \overset{N}{-y^2\sin(x)} \rangle$

Want $f(x,y)$ s.t. $f_x = M = x^2\cos(y)$ and $f_y = N = -y^2\sin(x)$

(1) Check $N_x = M_y$

$-y^2\cos(x) \neq -x^2\sin(y)$

DONE!

There is no potential function!

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Panel 18

$F(x,y) = \langle \overset{M}{x^2+y^2}, \overset{N}{\underline{2xy}} \rangle$

(1) $N_x = 2y$ $M_y = 2y$ ✓

(2) Need $f(x,y)$ s.t. $f_x = M$, $f_y = N$

$f_x = x^2+y^2 \Rightarrow f = \int x^2+y^2\,dx = \underline{\frac{1}{3}x^3 + xy^2 + C(y)}$

$f_y = \cancel{2xy} + C'(y) = \cancel{2xy} \Rightarrow C'(y) = 0 \Rightarrow C(y) = c$

$\Rightarrow \underline{f(x,y) = \frac{1}{3}x^3 + xy^2 + c}$

18

Panel 19

$$\begin{array}{ccc} M & N & P \end{array}$$

Find potential for $\vec{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$ if exists.

Want $f(x,y,z)$ s.t. $f_x = M,\ f_y = N,\ f_z = P$ — potential

① $\text{curl}(F) = 0$?

$f(x,y,z) = xy^2 + ye^{3z} + \cancel{D(z)}\ d$

② $f_x = y^2 \Rightarrow f = \int y^2\,dx = xy^2 + C(y,z)$

$f_y = 2\cancel{x}y + C_y(y,z) \overset{\text{want}}{=} 2\cancel{x}y + e^{3z} \Rightarrow C_y(y,z) = e^{3z}$

$\Rightarrow C(y,z) = ye^{3z} + D(z)$

$f_z = \cancel{3ye^{3z}} + D'(z) = \cancel{3ye^{3z}}$ $\quad D(z) = d$