Panel 1

Last Time: Integration in $\mathbb{R}^n$:

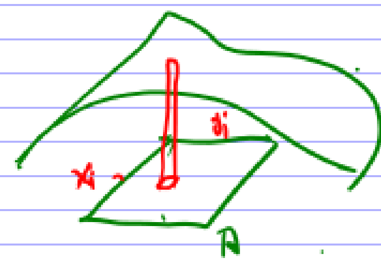
Volume under $z=f(x,y)$

$$\iint_A f(x,y)\,dA = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i,y_j)\,\Delta x_i\,\Delta y_j$$

Fubini's Thm (How to integrate in $\mathbb{R}^n$)

f cont. on $A=[a,b]\times[c,d]$. Then

$$\iint_A f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

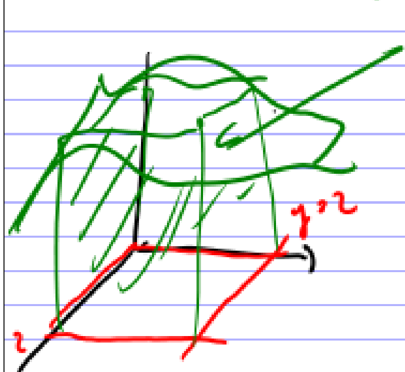


1

Panel 2

Ex: Find the volume of the solid bounded by $x^2+y^2+z=16$, the planes $x=2$ and $y=2$, and the coordinate planes.

$z = 16-x^2-y^2$



$$V = \iint_A 16-x^2-y^2\,dA = $$

$$= \int_0^2\int_0^2 16-x^2-y^2\,dx\,dy = $$

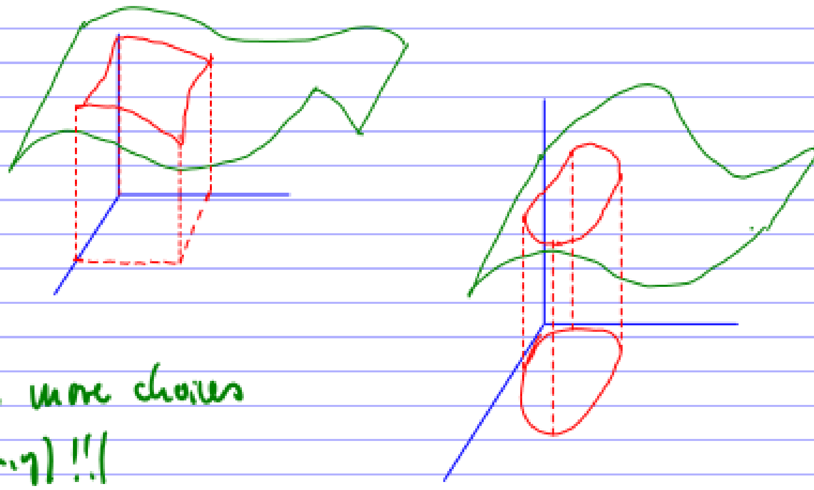
$= $ int(int($16-x^2-y^2$, $x=0..2$), $y=0..2$)

$\frac{160}{3}$

2

Panel 3

In $\mathbb{R}$ all we ever did was integrate over intervals $[a,b]$. In $\mathbb{R}^2$ it is different:



Many more choices for (x,y) !!!

3

Panel 4

Type 1 Region: $D = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$

Type 2 Region: $D = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$

Type 1:
$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type 2:
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

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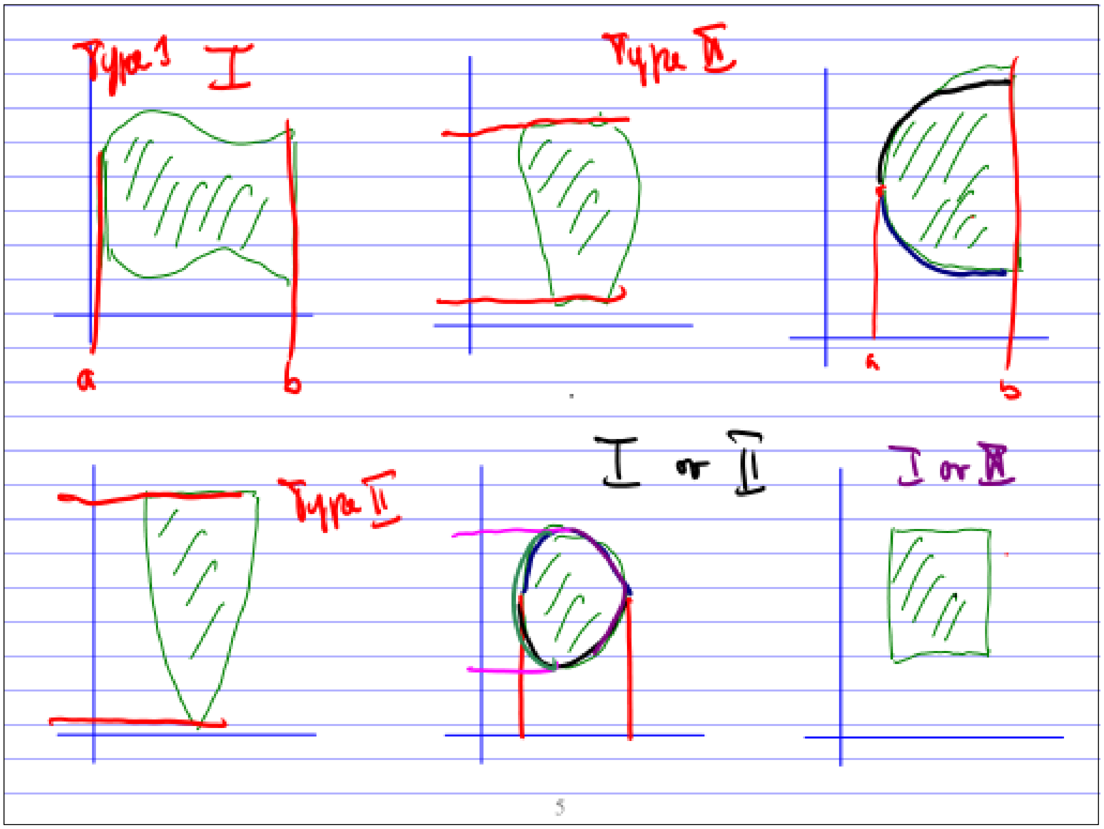
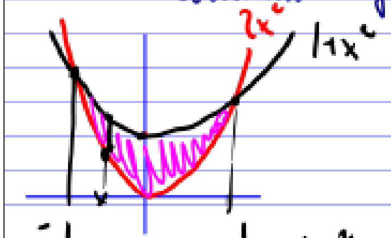
Panel 5



Panel 6

Ex: Find $\iint_D (x+2y)\,dA$ where D is the region bounded by $y=2x^2$ and $y=1+x^2$



$2x^2 = 1+x^2 \qquad x=\pm 1$

int(int(x·y + 2 y, y = 2 x^2 ..1 + x^2), x = -1 ..1)

$\frac{32}{15}$

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y \; dy \; dx =$$

$$\int_{-1}^{1} xy+y^2\Big|_{2x^2=y}^{1+x^2=y} dx = \int_{-1}^{1}\left[x(1+x^2)+(1+x^2)^2\right]-\left[x\,2x^2-(2x^2)^2\right]dx$$

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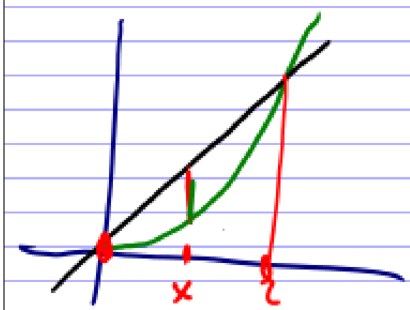
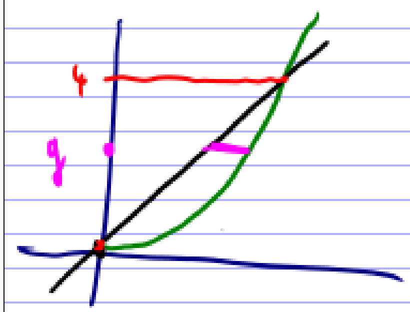
Panel 7

Ex: Volume under $z = x^2+y^2$ above $y=2x$ and $y=x^2$

$\frac{1}{2}y = x$ $\sqrt{y} = x$



$$\int_0^2 \int_{x^2}^{2x} x^2+y^2 \; dy \, dx$$



$$\int_0^4 \int_{\cancel{2x}\ \frac{1}{2}y}^{\cancel{x^2}\ \sqrt{y}} x^2+y^2 \; dx \, dy$$

comes out the same!!

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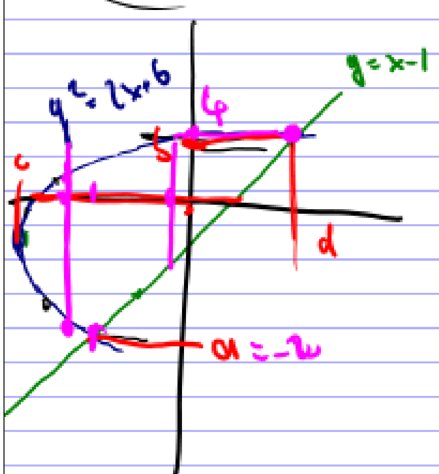
Panel 8

Ex: Find $\iint_D xy \, dA$ where D is bounded by $y=x-1$ and $y^2=2x+6$. Should you $\iint xy \, dx\,dy$ or $\iint xy \, dy\,dx$?



$$\int_{-2}^{4} \int_{\frac{1}{2}(y^2-6)}^{y+1} xy \; dx\,dy$$

$$\int_c^d \int xy \; dy\,dx$$ requires 2 integrals

$y = 4, -2$

$y^2 = 2x+6 \Rightarrow \frac{1}{2}(y^2-6) = x$

$y = x-1 \Rightarrow x = y+1$

$y+1 = \frac{1}{2}(y^2-6)$ $(y+2)(y-4)$

$2y+2 = y^2-6$ $0 = y^2-2y-8$

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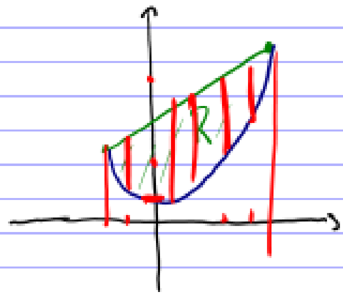
Panel 9

dx dy or dy dx ?

(A)

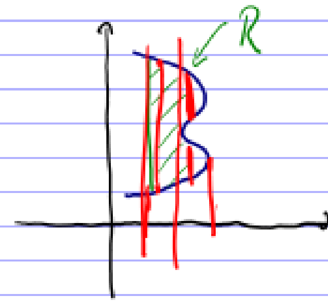


① $\iint_R f(x,y)\,dx\,dy$ 2 int

② $\iint_R f(x,y)\,dy\,dx$ 1 int.

① $\iint_R f(x,y)\,dx\,dy$ 1 int

② $\iint_R f(x,y)\,dy\,dx$ 3 int



(B)

3

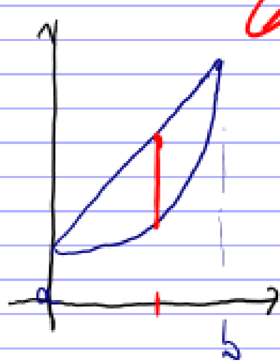
9

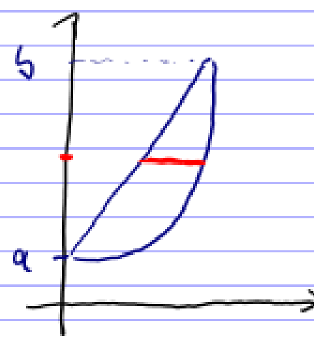
Panel 10

Which picture represents $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

1

(A)





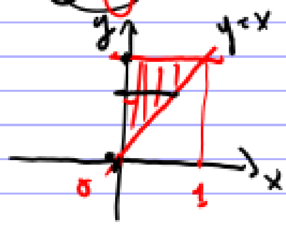
(B)

10

Panel 11

Occasionally there are other reasons that you would prefer one integration order over another - after all, you do have choices.

Ex: Find $\int_0^1\int_x^1 \sin(y^2)\,dy\,dx = \int_0^1\int_0^y \sin(y^2)\,dx\,dy$



y: y = x to y = 1

$= \int_0^1 x\sin(y^2)\Big|_{x=0}^{x=y} dy = \int_0^1 y\sin(y^2)\,dy$

Turns out: $\int \sin(x^2)\,dx$, $\int e^{x^2}\,dx$, $\int \cos(y^2)\,dy$ have no antiderivative!

Panel 12

$\int_0^1 y\sin(y^2)\,dy = \frac{1}{2}\int_0^1 \sin(u)\,du = \frac{1}{2}\left(-\cos(u)\right)\Big|_0^1 = -\frac{1}{2}\left(\cos(1)+1\right)$

Let $u = y^2$

$du = 2y\,dy$

$\cancel{dx}$

Panel 13

$$\int_a^b \int_c^d f(x,y)\, dA$$ easy - 2 choices, pick either one.

$$\int_a^b \int_{g_1(x)}^{h_1(y)} f(x,y)\, dy\, dx$$

$$\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\, dx\, dy$$

can be switched, but you need to carefully draw the bounds to switch them appropriately!

12:08

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Panel 14

$$\int_0^2 \int_0^1 x^2 y \, dx\, dy = \int_0^2 \frac{1}{3}x^3 y \Big|_{x=0}^{x=1} dy = \int_0^2 \frac{1}{3} y\, dy = \frac{1}{6}y^2 \Big|_0^2 = \frac{4}{6}$$

$$\int_0^1 \int_0^{\cancel{y}} \sqrt{1-y^2}\, dy\, dx$$ out!

$$\int_0^1 \int_0^y \sqrt{1-y^2}\, dx\, dy = \int_0^1 x\sqrt{1-y^2} \Big|_{x=0}^{x=y} dy = \int_0^1 y\sqrt{1-y^2}\, dy$$

Let $u = 1-y^2$

$du = -2y\,dy$

$$= -\frac{1}{2}\int_1^0 \sqrt{u}\, du = -\frac{1}{2}\cdot\frac{2}{3}\, u^{3/2}\Big|_1^0 = \frac{1}{3}$$

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Panel 15

Properties of Double Integrals

(1) $\iint_D f(x,y) + g(x,y)\, dA =$

(2) If $f(x,y) \geq g(x,y)$ then

(3) $\iint_D 1\, dA =$

(4) If $m \leq f(x,y) \leq M$ then

$\iint_D f(x,y)\, dA$

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Panel 16

Ex: Estimate $\iint_D e^{\sin(x)\cos(y)}\, dA$ where D is disk, radius 2

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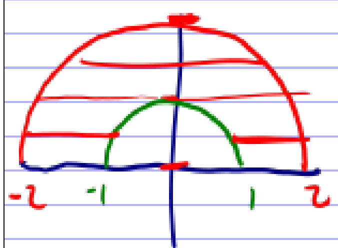
Panel 17

But there are some integral where all tricks (so far) don't work:

$\iint_D (3x + 4y^2)\, dA$ where D is region in upper half plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$



$\int_0^2 \int \quad dx\,dy$ 3int

$\int \int \quad dy\,dx$ 3int!

give up!

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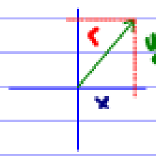
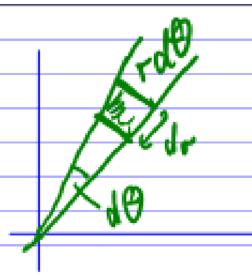
Panel 18

Solution. Polar Coordinates



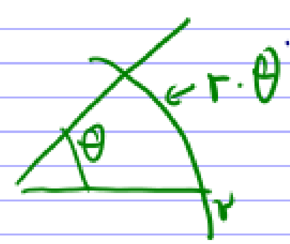
$x = r\cos\theta$

$y = r\sin\theta$



$dx\,dy = r\,dr\,d\theta$

Thus $\iint_A f(x,y)\, dA =$



$\iint_A f(x,y)\,dx\,dy = \iint_A f(r,\theta)\,\boxed{r}\,dr\,d\theta$

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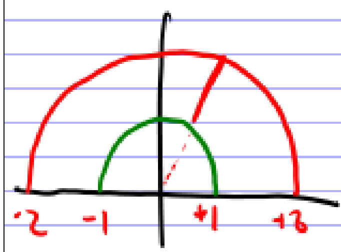
Panel 19

$$\iint_D 3x^2+3y^2\,dA$$, where D is the region in the upper half plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$



$x = r\cos\theta$

$y = r\sin\theta$

$$\iint_D 3x^2+3y^2\,dA =$$

$$\int_0^{\pi}\int_1^{2} 3r^2\cos^2\theta+3r^2\sin^2\theta\, r\,dr\,d\theta =$$

$$\int_0^{\pi}\int_1^{2} 3r^2 r\,dr\,d\theta = \int_0^{\pi}\int_1^{2} 3r^3\,dr\,d\theta = \int_0^{\pi} \frac{3}{4}r^4\Big|_1^2\,d\theta$$

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Panel 20

$$\int_0^{\pi}\int_1^{2} 3r^2 r\,dr\,d\theta = \int_0^{\pi}\int_1^{2} 3r^3\,dr\,d\theta = \int_0^{\pi} \frac{3}{4}r^4\Big|_1^2\,d\theta =$$

$$= \int_0^{\pi}\left(\frac{3}{4}16 - \frac{3}{4}\cdot 1\right)d\theta =$$

$$\int_0^{\pi}\left(12-\frac{3}{4}\right)d\theta = \left(12-\frac{3}{4}\right)\theta\Big|_0^{\pi} =$$

$$= \left(12-\frac{3}{4}\right)\cdot\pi$$

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Panel 21

Ex: $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

HW

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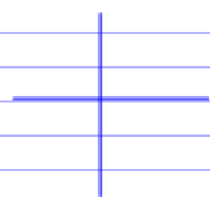
Panel 22

Ex: Volume under $z = x^2+y^2$, inside $x^2+y^2 = 2x$, above xy-plane



```
with(plots):
implicitplot3d({z = x^2 + y^2, x^2 + y^2 = 2x}, x = -2..2, y = -2..2, z = 0..4);
```

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Panel 23

Ex: Volume under $z = x^2 + y^2$, inside $x^2 + y^2 = 2x$, above xy-plane