Panel 1

Last Time:

$f_x(x-x_0)+f_y(y-y_0)+z_0 = z = p(x,y)$

Tangent Plane

PDE: subst. f into problem. Does it check?

$\|u\|=1$

Directional deriv. of f in dir. $\vec{u}$ is. $D_u(f) = (\nabla f)\cdot(u)$

Gradient and its Properties: $\nabla f = \langle f_x, f_y, f_z\rangle$

- The gradient is a vector
- Gradient is perpendicular to level curves
- Gradient points in direction of max. increase
- $\|\nabla f\|$ is the max. increase

1

Panel 2

Ex: Find ∇f if $f(x,y,z) = \ln(xy^2z^3)$

$$\langle f_x, f_y, f_z\rangle = \left\langle \frac{1}{xy^2z^3}\, y^2z^3,\ \frac{1}{xy^2z^3}\cdot 2xyz^3,\ \frac{1}{xy^2z^3}\, 3xy^2z^2 \right\rangle$$

$$= \frac{1}{xy^2z^3}\, yz^2\,(yz,\ xz,\ xy) = \frac{1}{xyz}\langle yz, 2xz, 3xy\rangle$$

Find max. rate of change for $f(x,y,z) = \ln(xy^2z^3)$ at $P(1,1,1)$

In which direction does this max. increase occur?

$$\|\nabla f\|_{(1,1,1)} = \|\langle 1,2,3\rangle\| = \underline{\sqrt{14}}$$

In dir. of ∇f, i.e. $\langle 1,2,3\rangle$

2

Panel 3

Ex: Suppose

$f(x,y) = y\ln(x)$. You are standing at $P(1,-3)$.
and you are heading in the direction $\langle -4,3\rangle$.
(1) Are you going up or down? (2) How much? (3) Which way should you go for max change in height? (4) What is the max change in height?

(1) $D_u(f) = \nabla f \cdot u =$, $u = \langle -5,6\rangle \frac{1}{\sqrt{61}}$.

$= \left\langle \frac{y}{x}, \ln(x)\right\rangle\Big|_{(1,-3)} \cdot u$

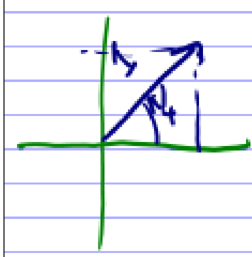
$= \langle -3, 0\rangle \cdot \langle -5,6\rangle \frac{1}{\sqrt{61}} = \frac{15}{\sqrt{61}}$ (2) goes up!

(3) $\langle -3, 0\rangle$ (4) 3

3

Panel 4

Which vector $\|u\|=1$ makes an angle of $\pi/4$



$x = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ $u = \langle \cos(\theta), \sin(\theta)\rangle$

$y = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f(x,y) = x^2y^3 - y^4$, $P(2,1)$, $u = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$

$D_{\vec{u}}(f) = \langle 2xy^3, 3x^2y^2 - 4y^3\rangle\big|_{(2,1)} \cdot \langle 1,1\rangle \cdot \frac{1}{\sqrt{2}}$

$\langle 4, 12-4\rangle \langle 1,1\rangle \frac{1}{\sqrt{2}} = (4+8)\frac{1}{\sqrt{2}} = \frac{12}{\sqrt{2}}$

4

Panel 5

$f = (x+2y+3z)^{7/2}$, $D_{\vec{u}}(f)$, $u = \langle 0, 2, -1\rangle \cdot \frac{1}{\sqrt{5}}$

$\frac{7}{2}(x+2y+3z)^{5/2}\langle 1, 2, 3\rangle \cdot \langle 0, 2, -1\rangle \frac{1}{\sqrt{5}} =$

$\frac{7}{2}(x+2y+3z)^{5/2}\, 1/\sqrt{5}$

5

Panel 6

$u_{tt} = a^2 u_{xx}$ $\qquad u(t,x) = \frac{t^2}{a^2t^2 - x^2}$

$$u_t = \frac{2t\cdot(a^2t^2-x^2) - t^2(2a^2t)}{(a^2t^2-x^2)^2} = \frac{-2x^2t}{(a^2t^2-x^2)^2}$$

$$u_{tt} = \frac{(-2x^2)(a^2t^2-x^2)^2 + 2x^2t\,2(a^2t^2-x^2)\,2a^2t}{(a^2t^2-x^2)^4}$$

$$= \frac{(a^2t^2-x^2)\left(-2x^2(a^2t^2-x^2) + 8a^2x^2t^2\right)}{(a^2t^2-x^2)^4} = \frac{6a^2x^2t^2+4x^4}{(a^2t^2-x^2)^3}$$

6

Panel 7

$$a(t,x) = \frac{t^2}{a^2t^2-x^2} = t^2(a^2t^2-x^2)^{-1}$$

$$a_x = -t^2(a^2t^2-x^2)^{-2}\cdot(-2x) = \frac{2xt^2}{(a^2t^2-x^2)^2}$$

$$a_{xx} = \frac{2t^2(a^2t^2-x^2)^2 + 2xt^2\,2\underline{(a^2t^2-x^2)}\,2x}{(a^2t^2-x^2)^4} =$$

$$= \frac{2t^2(a^2t^2-x^2)+8x^2t^2}{(a^2t^2-x^2)^3} = \frac{6x^2t^2+2a^2t^4}{(a^2t^2-x^2)^3}$$

Sorry!

Panel 8

$z = y\cdot\ln(x)$, $\quad P(1,4,0)$

$$\underline{4}(x-1) + \underline{0}(y-4) + \underline{0} = p(x,y)$$

$$f_x = \frac{y}{x}\Big|_{(1,4)} = 4 \qquad f_y = \ln(x)\Big|_{(1,4)} = 0$$

Panel 9

Name:

Quiz

① Consider $f(x,y)=x^2+3xy-y^2$. Find

a) $f_x = 2x+3y$

b) $\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = (f_x)_y = 3$

c) $\nabla f = \langle 2x+3y, 3x-2y \rangle$

② If $f(x,y) = xy+3xy^2$. Find tangent plane at $P(1,1,4)$

$4(x-1)+7(y-1)+4 = p(x,y)$

$f_x = y+3y^2 \big|_{(1,1)} = 4 \quad , f_y = x+6xy \big|_{(1,1)} = 7$

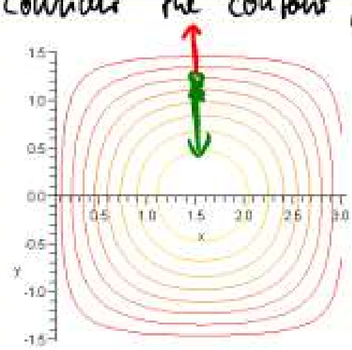
9

Panel 10

③ $f(x,y) = x^3-3xy+4y^2$. Find directional derivative in the direction of $\langle 3/5, 4/5 \rangle$

$\langle 3x^2-3y, 3x+8y \rangle \langle 3/5, 4/5 \rangle = \frac{1}{5}(9x^2-9y+12x+32y)$

④ Consider the contour plot below. Sketch the gradient at $P(1.5, 1.0)$

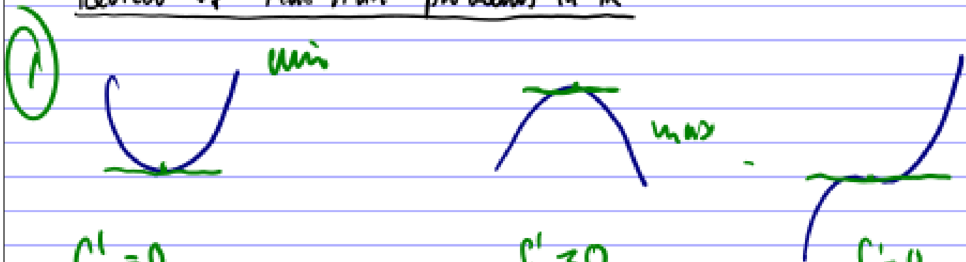


10

Panel 11

Review of Max/Min problems in $\mathbb{R}$

1



$f' = 0$ $\quad$ $f' = 0$ $\quad$ $f' = 0$

$f'' > 0$ $\quad$ $f'' < 0$. $\quad$ $f'' = 0$

1. Find f'
2. Solve $f' = 0$ critical points
3. Check f'': $f'' > 0$: min
   $f'' < 0$: max
   $f'' = 0$: no info

$f(x) = x^4$

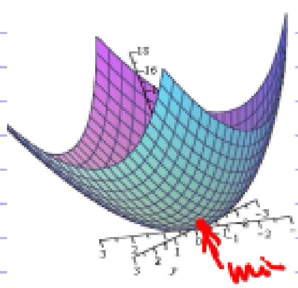
$f'(x) = 4x^3 = 0 \rightarrow \underline{x = 0}$

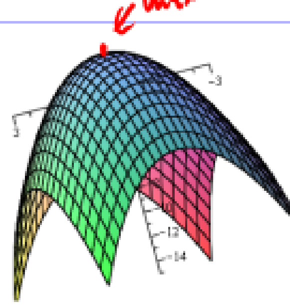
$f''(x) = 12x^2 \Rightarrow f''(0) = 0$

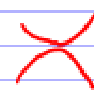
ROT

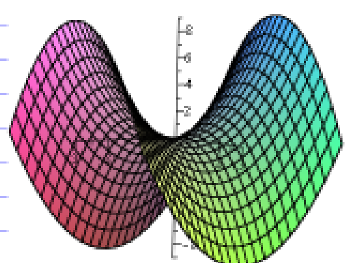
Panel 12

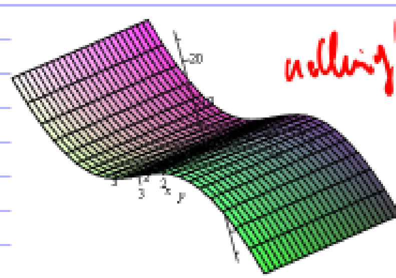
How about in $\mathbb{R}^2$











12

Panel 13

Max / Min Problems

To find max/min of $z = f(x,y)$:

1. Find ∇f
2. Solve $\nabla f = 0$ (system of equations)
3. Compute $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ and $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

a) f has min if: $D > 0$ and $f_{xx} > 0$
b) f has max if: $D > 0$ and $f_{xx} < 0$
c) f has saddle if: $D < 0$
d) no information if: $D = 0$

at critical point!

13

Panel 14

Ex: Find and classify the critical points for

$$f(x,y) = x^2 - 2xy + 3y^2 + 4x$$

(1) $f_x = 2x - 2y + 4 = 0$
(2) $f_y = -2x + 6y = 0$

(1) + (2): $4y + 4 = 0$

$-2x - 6 = 0$

$\Rightarrow y = -1,\ x = -3$ $\Rightarrow$ critical $(-3,-1)$

(4) $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}$

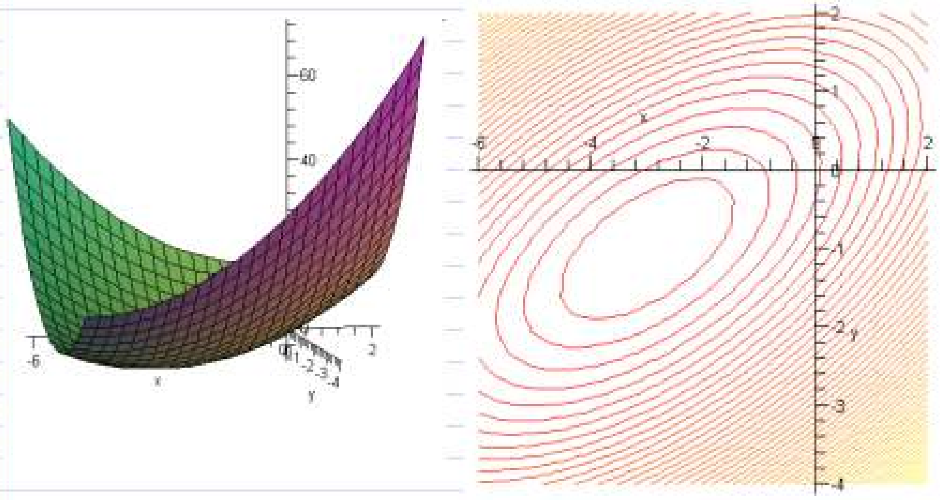
$D = 2 \cdot 6 - (-2)(-2) = 12 - 4 = 8 > 0$, $f_{xx} = 2 > 0$

Thus, $(-3,-1)$ is a local min!

14

Panel 15

$f(x,y) = x^2 - 2xy + 3y^2 + 4x$



15