Panel 1

Pi Day was last Week

3.14

1

Panel 2

Last time: Review of contour plots and surfaces.

limits: subst. $\to \frac{0}{0}$ try $x=0, y=0, x=y, \ldots$ No l'Hôpital prove it

continuity: limit problem

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} \quad (= f_{yx})$$ true in most cases but not always!

2

Panel 3

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

$$f(x,y)=\begin{cases} \dfrac{x^2y}{x^4+y^2}, & \text{if } (x,y)\neq(0,0) \\ \cancel{42}\ 3, & \text{if } (x,y)=(0,0) \end{cases}$$

Is f cont. at (0,0)

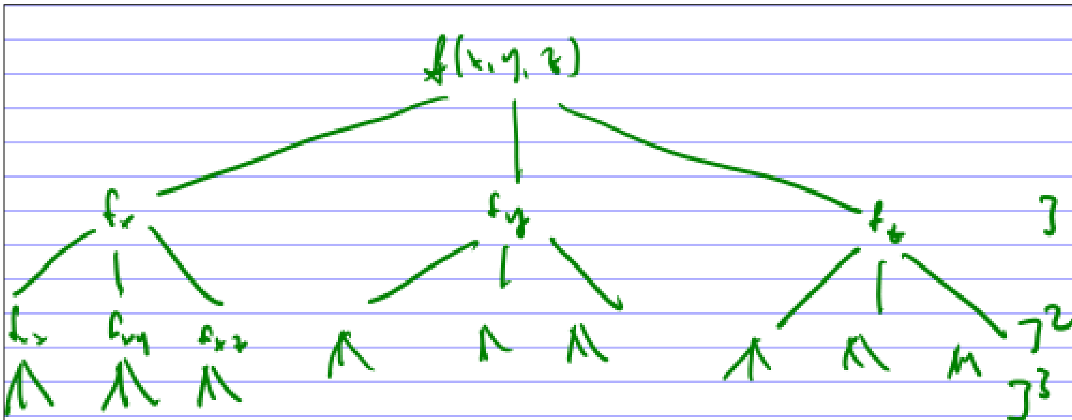
(1) $f(0,0) = \cancel{42}\ 3$

(2) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ dne. NOT $=3$

(3) (1) = (2)

3

Panel 4



$\Rightarrow$ n-th partial derivs.: 3^n partials (many of which agree)

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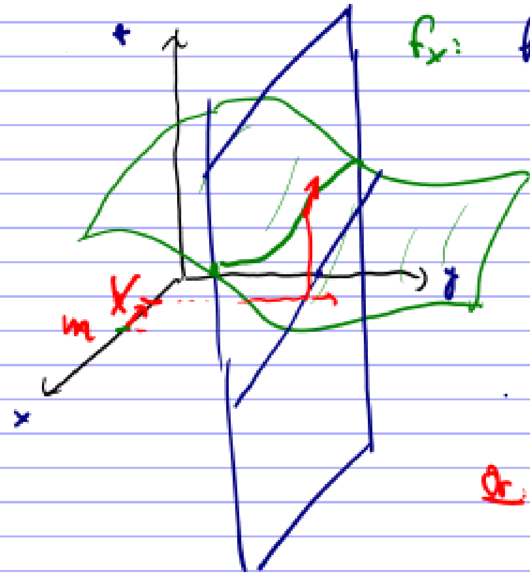
Panel 5

f_x: fix y (y=3)



f_x is slope of curve (f(x,y) intersection y = constant)

Or slope in the x-direction

f_y is slope in y-direction

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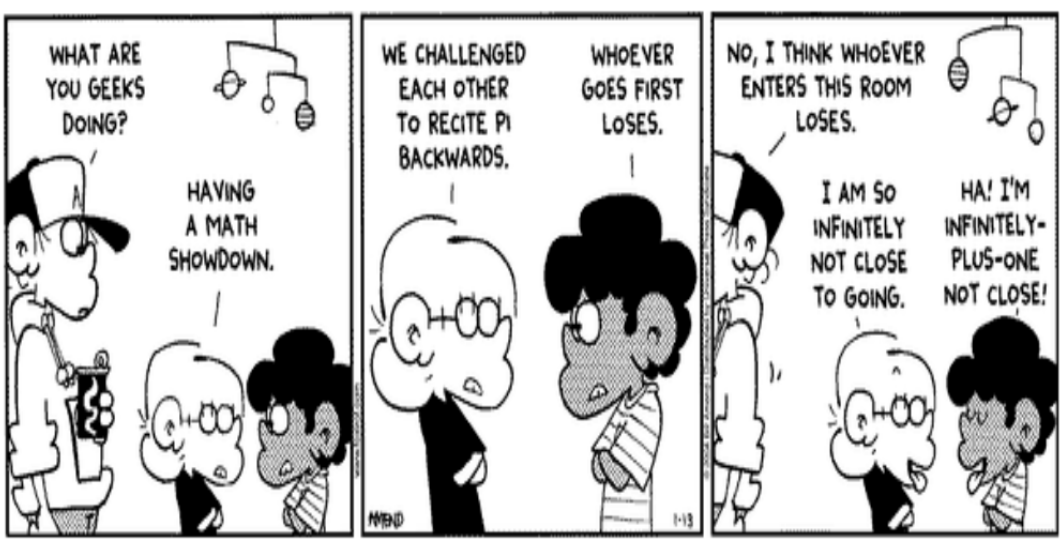
Panel 6



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Panel 7

Ex: Let $f(x,y,z) = xyz\sin(z)$. Find

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = (f_x)_y \qquad f_x = yz\sin(z)$$
$$f_{xy} = z\sin(z)$$

$$\frac{\partial^3 f}{\partial z \partial y^2} = f_{zyy}: \qquad f_z = xy\sin(z) + xyz\cos(z)$$
$$f_{zy} = x\sin(z) + xz\cos(z)$$
$$f_{zyy} = 0$$

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Panel 8

Partial derivatives frequently occur in Physics to describe laws of nature as PDEs (partial differential equations). For example: the Laplace PDE

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = u_{xx} + u_{yy} = 0$$

is important in heat conduction and fluid flow.

Ex: Show that $f(x,y) = e^x \sin(y)$ satisfies the above PDE

$$f_x = e^x \sin(y), \quad f_{xx} = e^x \sin(y)$$
$$f_y = e^x \cos(y), \quad f_{yy} = -e^x \sin(y)$$
$$\Rightarrow = 0$$

$x + y' = 7$ is DE

$f_{xx} + f_{yy} = 7$ is PDE

So f DOES solve Laplace PDE

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Panel 9

Ex: ~~If~~ $f(x,y) = \sin(x)\cosh(y)$ harmonic?

Recall. $\cosh(t) = \frac{1}{2}\left(e^{t} + e^{-t}\right)$ $\qquad \frac{d}{dt}\cosh(t) = \sinh(t)$

$\sinh(t) = \frac{1}{2}\left(e^{t} - e^{-t}\right)$ $\qquad \frac{d}{dt}\sinh(t) = \cosh(t)$

Harmonic means: solves Laplace PDE!

$f_x = \cos(x)\cosh(y)$ $\qquad f_{xx} = -\sin(x)\cosh(y)$

$f_y = \sin(x)\sinh(y)$ $\qquad f_{yy} = \sin(x)\cosh(y)$

YES: f is harmonic i.e. solves Laplace PDE

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Panel 10

f_x = slope in x-dir

f_y = slope in y-dir

$f(x)$: 1-variable: graph has tangent line

$f(x,y)$: 2-vars: graph has tangent plane

Tangent plane: $\frac{a}{c}(x-x_0)+\frac{b}{c}(y-y_0)+\cancel{c}(z-z_0)=0$

$$\Rightarrow z = A(x-x_0) + B(y-y_0) + z_0 = \ell(x,y)$$

$\frac{\partial}{\partial x}\ell(x,y) = A = f_x$

$\frac{\partial}{\partial y}\ell(x,y) = B = f_y$

$\boxed{z = f_x(x-x_0) + f_y(y-y_0) + z_0 \text{ is equation of } \underline{\text{tangent plane}}}$

Def.

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Panel 11

Equation of tangent plane to $f(x,y)$ at (x_0,y_0) is:

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

Ex: $f(x,y) = 2x^2+y^2$. Find tangent plane at $P(1,1,3)$

$f_x = 4x$ at $(1,1)$ $\Rightarrow$ $f_x = 4$

$f_y = 2y$ $\qquad$ $f_y = 2$

$z = 4(x-1) + 2(x-1) + 3$

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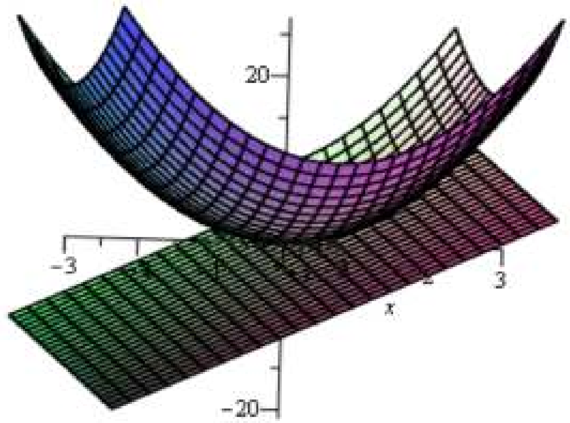
Panel 12

$f(x,y) = 2x^2+y^2$ Tangent plane at $P(1,1,3)$ is:

$z = 4(x-1) + 2(y-1) + 3$

$plot3d(\{2\cdot x^2 + y^2, 4\cdot(x-1) + 2\cdot(y-1) + 3\}, x=-3..3, y=-3..3)$



$f_x = 4$

$f_y = 2$

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Panel 13

Directional Derivatives:

f_x is deriv in x-dir

f_y is deriv. in y-dir

What about deriv in dir of $\vec{v} = \langle v_x, v_y \rangle$

Def: Direction deriv. of f in dir. $\vec{v} = \langle v_x, v_y \rangle$, v must be unit vector, is

$$\lim_{h \to 0} \frac{f(x+hv_x, y+hv_y) - f(x,y)}{h} = D_{\vec{v}} f(x,y)$$

Note: $D_{\langle 1,0 \rangle} f = f_x$

$D_{\langle 0,1 \rangle} f = f_y$

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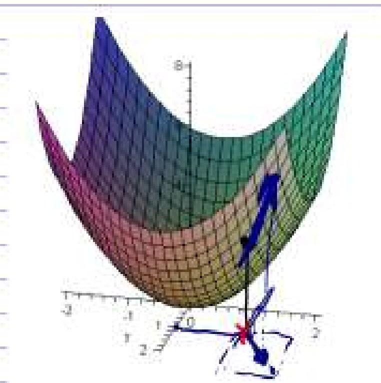
Panel 14

Ex: $f(x,y) = x^2 + y^2$

Find directional derivative in direction of $\vec{u} = \frac{1}{\sqrt{2}} \langle 1,1 \rangle$ at $(1,1)$



Note: $\|\vec{u}\| = \frac{1}{\sqrt{2}} \|\langle 1,1 \rangle\| = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(x+h\frac{1}{\sqrt{2}}, y+h\frac{1}{\sqrt{2}}) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+\frac{h}{\sqrt{2}})^2 + (y+\frac{h}{\sqrt{2}})^2 - (x^2+y^2)}{h} =$$

$D_{\vec{u}} f(1,1) =$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + \frac{2}{\sqrt{2}}xh + \frac{h^2}{2} + \cancel{y^2} + \frac{2}{\sqrt{2}}yh + \frac{h^2}{2} - \cancel{x^2} - \cancel{y^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(\frac{2}{\sqrt{2}}x + \cancel{\frac{h}{2}}^{\,0} + \frac{2}{\sqrt{2}}y + \cancel{\frac{h}{2}}^{\,0})}{\cancel{h}} = \frac{2}{\sqrt{2}}(x+y) = 4/\sqrt{2} = 2\sqrt{2}$$

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Panel 15

<u>Thm:</u> Suppose $\vec{u} = \langle a, b \rangle$ is a unit vector. Then

$$D_{\vec{u}} f(x,y) =$$

<u>Ex:</u> $f(x,y) = x^2 + y^2$, $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1 \rangle$. Find $D_{\vec{u}} f(x,y)$ at $(1,1)$

$$D_{\vec{u}} f = \langle f_x, f_y \rangle \cdot \langle u_x, u_y \rangle =$$

$$= \langle 2x, 2y \rangle \cdot \frac{1}{\sqrt{2}}\langle 1,1 \rangle = \frac{1}{\sqrt{2}}\left(2x \cdot 1 + 2y \cdot 1\right) = \frac{2}{\sqrt{2}}(x+y)$$

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Panel 16

<u>Ex:</u> $f(x,y) = x^3 - 3xy + 4y^2$. Find directional derivative in the direction of $\langle \cos(\pi/6), \sin(\pi/6) \rangle$

$$D_u(f) = \lim_{h \to 0} \frac{f(x + h \cdot \cos(\pi/6), y + h \sin(\pi/6)) - f(x,y)}{h}$$

<u>Better Way:</u> <u>Thm:</u> $D_{\vec{u}} f = f_x \cdot u_x + f_y u_y =$

$$= \langle f_x, f_y \rangle \cdot \langle u_x, u_y \rangle$$

$$\Rightarrow D_{\vec{u}} f = \langle \underbrace{3x^2 - 3y}_{f_x}, \underbrace{-3x + 8y}_{f_y} \rangle \cdot \langle \underbrace{\tfrac{\sqrt{3}}{2}}_{\cos(\pi/6)}, \underbrace{\tfrac{1}{2}}_{\sin(\pi/6)} \rangle = \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y)$$

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Panel 17

Def: If $f(x,y)$ is a function of 2 variables, then the gradient of f is: $\nabla f = (f_x, f_y)$ gradient

$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$

Note: $D_{\vec{u}} f = \langle f_x, f_y \rangle \cdot (u_x, u_y) = (\nabla f) \cdot u$

Ex: $f(x,y) = xy \sin(x)$.

$\nabla f = \langle y \cdot \sin(x) + xy \cos(x),\ x \sin(x) \rangle$

$f(x,y,z) = x \cos(y^2+z^2)$

$\nabla f = \langle \cos(y^2+z^2), -x \sin(y^2+z^2) \cdot 2y, -x \sin(y^2+z^2)\,2z \rangle$

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Panel 18

Recall: $D_{\vec{u}} f = \nabla f \cdot u = \|\nabla f\| \, \|u\| \cos(\theta) = \|\nabla f\| \cos\theta$ fixed

Theorem: The max. value of $D_{\vec{u}}(f)$ is $\|\nabla f\| \cdot 1$ and is attained if $\vec{u}$ points in direction of ∇f

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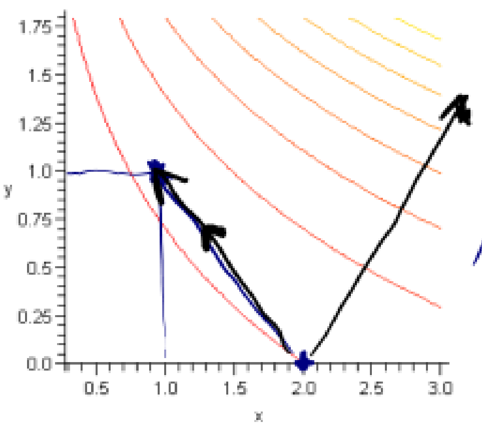
Panel 19

Ex: $f(x,y) = xe^y$. Find rate of change at $\underline{P(2,0)}$ in the direction from P to $Q(1,1)$.

$$D_u f(2,0) = (\nabla f)\cdot(u)$$

$$\vec{u} = \frac{1}{\|PQ\|}PQ = \frac{1}{\sqrt{2}}\langle -1,1\rangle$$



↗ up

$$\nabla f = \langle f_x, f_y\rangle = \langle e^y, xe^y\rangle$$

$$\Rightarrow \nabla f\big|_{P(2,0)} = \langle 1,2\rangle$$

$$D_{\vec{u}} f(2,0) = \langle 1,2\rangle\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle = \frac{1}{\sqrt{2}}(-1+2) = \underline{\frac{1}{\sqrt{2}}}$$

Dir of largest incr.

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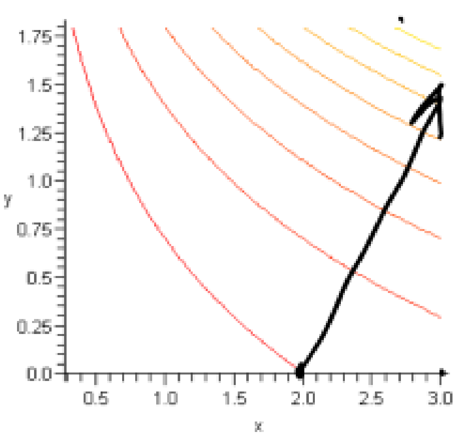
Panel 20

Ex: $f(x,y) = xe^y$. You are standing at $P(2,0)$. In which direction does f change most rapidly, and what is that rate of change?



$$D_u f = (\nabla f, u) = \|\nabla f\|\,\|u\|\cos\theta$$

$$= \|\nabla f\|\cos\theta$$

is largest if angle $\theta = 0$,

i.e. $\vec{u}$ points in ∇f dir.

and has value $\|\nabla f\|$

$$\nabla f = \langle 1,2\rangle,\ \|\nabla f\| = \sqrt{5}$$

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Panel 21

Summary: If $f(x,y)$ is a function then

$\nabla f = \langle f_x, f_y \rangle$ is the gradient

(a) ∇f is a vector

(b) Gradient is perp to level curves

(c) Gradient points in direction of steepest increase

(d) Max rate of change is $\|\nabla f\|$

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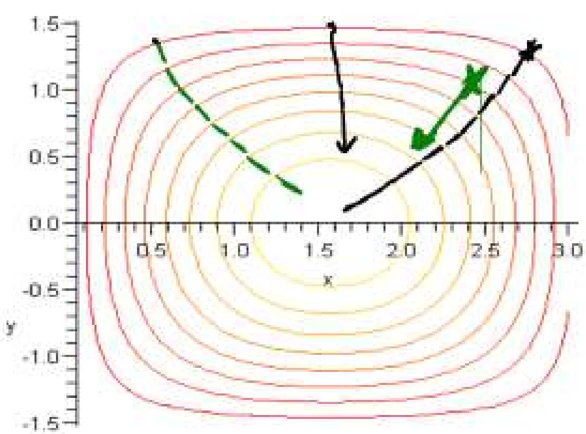
Panel 22

Below is a contour plot for $f(x,y)$, showing several level curves. Sketch ∇f at $P(2.5, 1.0)$, approx.



Note: Q7 on Wat.

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