Higher Order Partial Derivatives

Note Title 3/22/2013

For $f(x,y)$ we defined partial derivatives

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

These are definitions, and you most certainly need to know them, but to actually find f_x you consider the other variable y as constant and apply diff. rules as usual.

Ex: If $f(x,y) = 2x\sin(x^2+y^2)$

$$\Rightarrow f_x(x,y) = 2\cdot\sin(x^2+y^2) + 2x\cos(x^2+y^2)\cdot 2x$$

Similarly, for f_y we keep x constant:

$$\Rightarrow f_y(x,y) = 2x\cdot\cos(x^2+y^2)\cdot 2y$$

Thus, partial derivatives are again functions of x,y, so we can take partials again.

Ex: If $f(x,y)$ is a 2D function, there are:

2 1^{st}-order partials: f_x, f_y

4 2^{nd}-order partials: f_{xx}, f_{xy}, f_{yx}, f_{yy}

8 3^{rd}-order partials: $f_{xxx}, f_{xxy}, f_{xyx}, f_{xyy}, f_{yyx}, f_{yyy}$

16 4^{th}-order, 32 5^{th}-order, etc.

Ex: Find all second order partials of

$$f(x,y) = 3xy^2 + 7x^3y$$

$$\Rightarrow \quad f_x(x,y) = 3y^2 + 21x^2y$$

$$\Rightarrow \quad f_{xx}(x,y) = 42xy$$

$$f_{xy}(x,y) = 6y + 21x^2$$

and $f_y(x,y) = 6xy + 7x^3$

$$\Rightarrow \quad f_{yx}(x,y) = 6y + 21x^2$$

$$f_{yy}(x,y) = 6x$$

Note that $f_{xy} = f_{yx}$ so that the order of differentiation does not seem to matter in this example. let's try a 3D ex:

Ex: Find all 9 2nd-order partials for

$$f(x,y,z) = x^2 + yz^2 - 5xyz + zx$$

$\Rightarrow f_x = 2x - 5yz + z$

$f_{xx} = 2$

$f_{xy} = -5z$

$f_{xz} = -5y + 1$

$\Rightarrow f_y = z^2 - 5xz$

$f_{yx} = -5z$

$f_{yy} = 0$

$f_{yz} = 2z - 5x$

$\Rightarrow f_z = 2yz - 5xy + x$

$f_{zx} = -5y + 1$

$f_{zy} = 2z - 5x$

$f_{zz} = 2y$

Note: $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$

This is not always true, but true for "nice" f.

Thm: If $f(x,y)$ is a function s.t. all 2^{nd}-order partials exist and cont., then $f_{xy} = f_{yx}$. A similar statement is true in $\mathbb{R}^3$

Thus, in most practical cases it is true!

Notation: Higher order deriv. are written as:

$$f_x = \frac{\partial}{\partial x} f, \quad f_y = \frac{\partial}{\partial y} f$$

$$f_{xx} = \frac{\partial^2}{\partial x^2} f, \quad f_{xy} = \frac{\partial^2}{\partial x \partial y} f,$$

$$f_{yx} = \frac{\partial^2}{\partial y \partial x} f, \quad f_{yy} = \frac{\partial^2}{\partial y^2} f$$

Ex: $f(x,y) = xe^{2xy^2}$. Find

$$\frac{\partial^2}{\partial x \partial y} = f_{xy} = (f_x)_y = \left(e^{2xy^2} + xe^{2xy^2} \cdot 2y^2\right)_y =$$

$$= 4xye^{2xy^2} + 4xye^{2xy^2} + 2xy^2 e^{2xy^2} \cdot 4xy$$

$$= 8xy e^{2xy^2} + 8x^2y^3 e^{2xy^2}$$

We can of course compute partials with Maple or Wolfram Alpha:

<u>Maple:</u>

Defining the function, using two variables as input:

$$f(x, y) := x \cdot \exp(2\, x \cdot y^2)$$

$$(x, y) \to x\, e^{2\, x\, y^2}$$

Taking one derivative with respect to x:

$diff(f(x, y), x)$

$$e^{2\, x\, y^2} + 2\, x\, y^2\, e^{2\, x\, y^2}$$

Taking an x derivative, then a y derivative:

$diff(f(x, y), x, y)$

$$8\, x\, y\, e^{2\, x\, y^2} + 8\, x^2\, y^3\, e^{2\, x\, y^2}$$

<u>WA:</u>