

Higher Order Partial Derivatives

Note Title

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For $f(x, y)$ we defined partial derivatives

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

These are definitions, and you **most certainly** need to know them, but basically
find f_x you consider the other variable y
as constant and apply diff. rules as usual.

Ex: If $f(x, y) = 2x \sin(x^2 + y^2)$

$$\Rightarrow f_x(x, y) = 2 \cdot \sin(x^2 + y^2) + 2x \cos(x^2 + y^2) \cdot 2x$$

Similarly, for f_y we keep x constant:

$$\Rightarrow f_y(x, y) = 2x \cdot \cos(x^2 + y^2) \cdot 2y$$

Thus, partial derivatives are again functions
of x, y , so we can take partials again.

Ex: If $f(x,y)$ is a 2D function, there are:

2 1st-order partials: f_x, f_y

4 2nd-order partials: $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

8 3rd-order partials: $f_{xxx}, f_{xxy}, f_{xyx}, f_{xyy}, f_{yyy}, f_{yyx}$

16 4th-order, 32 5th-order, etc.

Ex: Find all second order partials of

$$f(x,y) = 3xy^2 + 7x^3y$$

$$\Rightarrow f_x(x,y) = 3y^2 + 21x^2y$$

$$\Rightarrow f_{xx}(x,y) = 42xy$$

$$f_{xy}(x,y) = 6y + 21x^2$$

and $f_y(x,y) = 6xy + 7x^3$

$$\Rightarrow f_{yx}(x,y) = 6y + 21x^2$$

$$f_{yy}(x,y) = 6x$$

Note that $f_{xy} = f_{yx}$ so that the order of differentiation does not seem to matter

in this example. Let's say a 3D ex:

Eg: Find all 9 2nd-order partials for

$$f(x, y, z) = x^2 + yz^2 - 5xyz + zx$$

$$\Rightarrow f_x = 2x - 5yz + z$$

$$f_{xx} = 2$$

$$f_{xy} = -5z$$

$$f_{xz} = -5y + 1$$

$$\Rightarrow f_{yz} = z^2 - 5xz$$

$$f_{yx} = -5z$$

$$f_{yy} = 0$$

$$f_{yz} = 2z - 5x$$

$$\Rightarrow f_z = 2yz - 5xy + x$$

$$f_{zx} = -5y + 1$$

$$f_{zy} = 2z - 5x$$

$$f_{zz} = 2y$$

Note: $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$

This is not always true, but true for "nice" f .

Thus, if $f(x, y)$ is a function s.t.

all 2nd-order partials exist and cont.,

then $f_{xy} = f_{yx}$. A similar statement
is true in \mathbb{R}^3 .

Thus, in most practical cases it is true!

Notation: Higher order deriv. are written as:

$$f_x = \frac{\partial}{\partial x} f, \quad f_y = \frac{\partial}{\partial y} f$$

$$f_{xx} = \frac{\partial^2}{\partial x^2} f, \quad f_{xy} = \frac{\partial^2}{\partial x \partial y} f,$$

$$f_{yx} = \frac{\partial^2}{\partial y \partial x} f, \quad f_{yy} = \frac{\partial^2}{\partial y^2} f$$

\Rightarrow $f(x, y) = xe^{2xy^2}$. Find

$$\frac{\partial^2}{\partial x \partial y} f_{xy} = (f_x)_y = \left(e^{2xy^2} + x e^{2xy^2} \cdot 2y^2 \right)_y =$$

$$= 4xye^{2xy^2} + 4xye^{2xy^2} + 2xy^2 e^{2xy^2} \cdot 4xy$$

$$= 8xy e^{2xy^2} + 8x^2y^2 e^{2xy^2}$$

We can of course compute partials with Maple or Wolfram Alpha:

Maple:

Defining the function, using two variables as input:

$$f(x, y) := x \cdot \exp(2x \cdot y^2)$$

$$(x, y) \rightarrow x e^{2xy^2}$$

Taking one derivative with respect to x:

$$\text{diff}(f(x, y), x)$$

$$e^{2xy^2} + 2xy^2 e^{2xy^2}$$

Taking an x derivative, then a y derivative:

$$\text{diff}(f(x, y), x, y)$$

$$8xy e^{2xy^2} + 8x^2y^3 e^{2xy^2}$$

WA:

The screenshot shows the WolframAlpha search interface. At the top, the logo "WolframAlpha" is visible with the tagline "computational knowledge engine". Below the logo, there is a search bar containing the input text: "diff(x*exp(2*x*y^2), x, y)". Below the search bar, there are several small icons for different functions. To the right of the search bar, there are buttons for "Example" and "See full answer". The main result area shows the derivative calculation under the heading "Derivative". The result is displayed as a mathematical equation: $\frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} (x \exp(2xy^2)) \right) = 8xy e^{2xy^2} (xy^2 + 1)$. There are also buttons for "Approximate form" and "Show step-by-step solution".