Panel 1

Last time

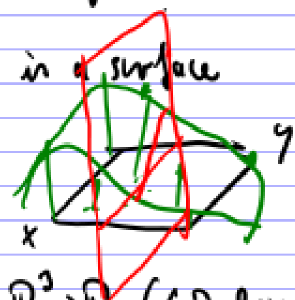
$f: \mathbb{R}^2 \to \mathbb{R}$ eg. $f(t,s) = \cos(s)\cdot\sin(t) = z$

$f: \mathbb{R}^3 \to \mathbb{R}$ eg. $f(x,y,z) = x\cdot y\cdot z$

Graphs + Slices { fix x, fix y — $f: \mathbb{R}^2 \to \mathbb{R}$ is a surface

Level Curves

Contour Plots

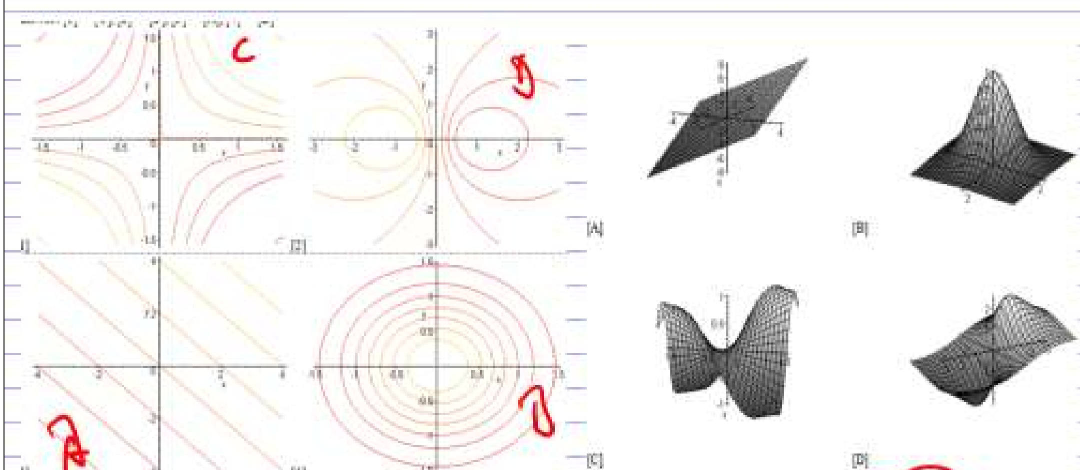


Graph of $f: \mathbb{R}^3 \to \mathbb{R}$ (4D objects)

Limits

1

Panel 2



Match Contour plots to Graphs

Q:1

2

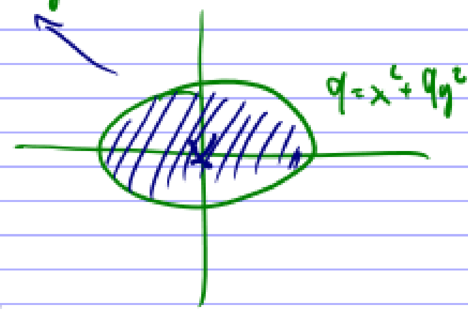
Panel 3

$\ln(\underbrace{9 - x^2 - 9y^2})$ domain:

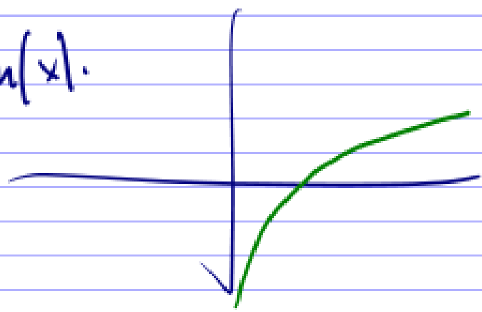
$$9 - x^2 - 9y^2 > 0$$

$$9 > x^2 + 9y^2$$

$9 = x^2 + 9y^2$ is ellipse



$\ln(x)$:



3

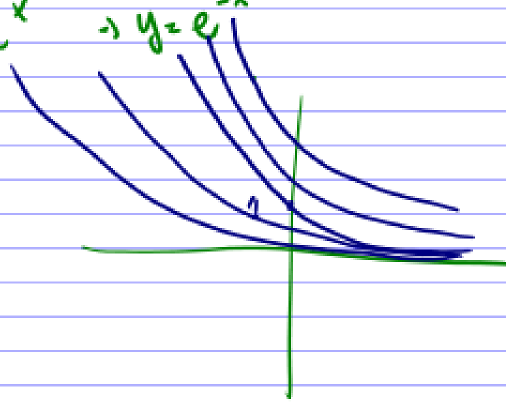
Panel 4

$z = ye^x$ level curves

$c = ye^x$ e.g. $1 = ye^x \rightarrow y = e^{-x}$

$c = ye^x$

$y = ce^{-x}$



4

Panel 5

<u>Limits</u> $f:\mathbb{R}^2\to\mathbb{R}$ $\quad \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=L$

<u>Def:</u> Given any $\varepsilon>0$ there is a $\delta>0$ such that

whenever $\|(x,y)-(x_0,y_0)\|<\delta$ then $|f(x,y)-L|<\varepsilon$

<u>How to really find limits</u>

1.) Subst (x_0,y_0) and hope for the best!

2.) fix $x=x_0,\ y\to y_0$
$y=y_0,\ x\to x_0$
$x=y$
$x=y^2,\ y=x^2$

if any of these are different $\Rightarrow$ Limit d.n.e. ☹

3) limit <u>might</u> exist $\Rightarrow$ prove it! (Hard)

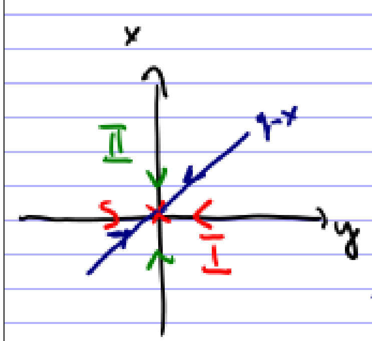
5

Panel 6

<u>Ex:</u> $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy}{x^2+y^2} = \dfrac{1}{2}$

d.n.e.

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \dfrac{0}{0}$ i.e. more work



I: $y=0,\ x\to 0$: $\lim\limits_{(x,0)\to(0,0)} \dfrac{x\cdot 0}{x^2+0^2} = 0$

II: $x=0,\ y\to 0$ $\lim\limits_{(0,y)\to(0,0)} \dfrac{0}{y^2} = 0$

III: $y=x\to 0$ $\lim\limits_{x\to 0} \dfrac{x^2}{x^2+x^2} = \lim\limits_{x\to 0} \dfrac{\cancel{x^2}}{2\cancel{x^2}} = \dfrac{1}{2}$

6

Panel 7

Ex: Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ if it exists Nope

$x=0, y\to 0$. $\lim \dfrac{0}{y^4} = 0$

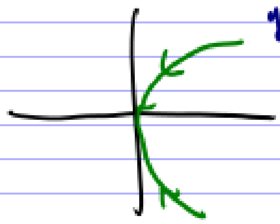
$y=0, x\to 0$: $\lim 0/x^2 = 0$

$x=y\to 0$

$$\lim_{x\to 0} \frac{x^3}{x^2+x^4} = \lim_{x\to 0} \frac{x^3}{x^2(1+x^2)} = \lim_{x\to 0} \frac{x}{1+x^2} = 0$$

$x=y^2$.

$$\lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$



7

Panel 8

Ex. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = \dfrac{0}{0}$

$x=0, y\to 0$: 0

$y=0, x\to 0$: 0

$y=x$ $\quad \lim\limits_{x\to 0} \dfrac{3x^3}{2x^2} = \lim\limits_{x\to 0} \dfrac{3}{2}x = 0$

$x=y^2$: $\quad \lim\limits_{y\to 0} \dfrac{3y^5}{y^4+y^2} = 0$ (l'Hospital works)

$y=x^2$: $\quad \lim \dfrac{3x^4}{x^2+x^4} = \lim \dfrac{3x^{\cancel{4}\,2}}{\cancel{x^2}(1+x^2)} = 0$

8

Panel 9

Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

Using different approach paths can only show that a limit does not exist!!

To prove a limit exists requires the def. (unfortunately)

Take $\varepsilon > 0$ there is $\delta > 0$ s.t. if $\|(x,y)-(0,0)\| < \delta$
$\Rightarrow |f(x,y) - 0| < \varepsilon$

$\left|\frac{3x^2y}{x^2+y^2}\right| < \varepsilon$ if $\sqrt{x^2+y^2} < \delta$

Know $x^2 \le x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} \le 1 \Rightarrow \frac{3x^2}{x^2+y^2} \le 3 \Rightarrow \frac{3x^2|y|}{x^2+y^2} \le 3|y| \le$

9

Panel 10

$\left|\frac{3x^2y}{x^2+y^2}\right| < \varepsilon$ if $\sqrt{x^2+y^2} < \delta$

Know $x^2 \le x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} \le 1 \Rightarrow \frac{3x^2}{x^2+y^2} \le 3 \Rightarrow \frac{3x^2|y|}{x^2+y^2} \le 3|y|$

But $|y| = \sqrt{y^2} \le \sqrt{x^2+y^2}$

Now $\frac{3x^2|y|}{x^2+y^2} \le 3\sqrt{x^2+y^2} = 3 \cdot \varepsilon/3 = \varepsilon \quad \Rightarrow \sqrt{x^2+y^2} \le \frac{\varepsilon}{3} = \delta$

Proof: Take $\varepsilon > 0$. I pick $\delta = \varepsilon/3$. Now if

$0.001 = \varepsilon$
$0.000333 = \delta$

$\sqrt{x^2+y^2} < \delta \Rightarrow \sqrt{x^2+y^2} < \frac{\varepsilon}{3}$

$|f(x,y)| \le 3\sqrt{x^2+y^2} < 3 \cdot \varepsilon/3 = \varepsilon$

10

Panel 11

Ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ ☹

$x=0,\ y\to 0:\ \lim \frac{0}{y^2}=0$ #

$y=0,\ x\to 0:\ \lim \frac{x^2}{x^2}=1$

\#

$x=0,\ y\to 0:\ \lim \quad 0$

$y=0,\ x\to 0:\ \lim \quad 0$

Ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$

$y=x,\ \to 0:\ \lim \frac{2x^3}{x^4+x^2}=0$

$x=y^2$

$y=x^2:\ \displaystyle\lim_{x\to 0} \frac{2x^4}{x^4+x^4}=1$

Ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}=$

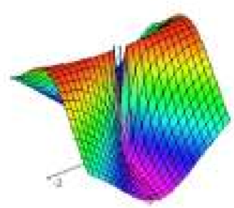
$x=0,\ y\to 0$
$y=0,\ x\to 0$
$y=x\to 0$
$y=x^2$
$x=y^2$

lim is 0!

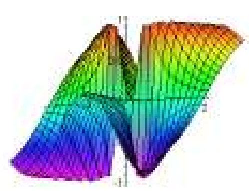
11

Panel 12

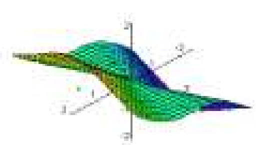
$f(x,y)=\dfrac{x^2}{x^2+y^2}$



$f(x,y)=\dfrac{2x^2y}{x^4+y^2}$



$f(x,y)=\dfrac{x^3}{x^2+y^2}$



$\boxed{\text{Prove this}}$

12

Panel 13

Prove $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$

$\left|\frac{x^3}{x^2+y^2}\right| = \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1} |x| \leq |x| \leq \sqrt{x^2+y^2}$ scrap paper

Given $\varepsilon > 0$, pick $\delta = \varepsilon$. Then, if

$\|(x,y)\| < \delta$

$\sqrt{x^2+y^2} < \varepsilon$

$\Rightarrow \left|\frac{x^3}{x^2+y^2}\right| \leq \sqrt{x^2+y^2} < \varepsilon \Rightarrow$

$\Rightarrow |f(x,y)| < \varepsilon$ #

13

back 12:05

Panel 14

Continuity

As usual, continuity is just rephrased limit question!

Def. $f(x,y)$ is continuous at (x_0,y_0) if

(1) $f(x_0,y_0)$ exists

(2) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exist

(3). (1)=(2)

Ex: $f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ \cancel{2}\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ ~~Not~~ yes see below

f cont. at (0,0)? $f(0,0) = 2 \neq \lim_{(x,y)\to(0,0)} f(x,y) \overset{?}{=} 0$

14

Panel 15

Derivatives: $\frac{d}{dx}$

If $f(x,y)$ is a function of 2 variables, define

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x} = f_x \quad \text{partial deriv w. resp. to } x$$

$$\lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \frac{\partial f}{\partial y} = f_y \quad \text{partial deriv w. resp. to } y$$

Same in $\mathbb{R}^3$: $\frac{\partial f}{\partial x} = f_x$

$\frac{\partial f}{\partial y} = f_y$

$\frac{\partial f}{\partial z} = f_z$

15

Panel 16

Ex: Find f_x if $f(x,y) = x^2y + y^2$

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0} \frac{[(x+h)^2y+y^2]-[x^2y+y^2]}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2y} + 2xhy + h^2y + \cancel{y^2} - \cancel{x^2y} - \cancel{y^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2xy + hy)}{\cancel{h}} = \underline{2xy}$$

$$f_y = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0} \frac{[x^2(y+h)+(y+h)^2]-[x^2y+y^2]}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2y} + \cancel{x^2}h + \cancel{y^2} + 2yh + h^2 - \cancel{x^2y} - \cancel{y^2}}{h} = \lim_{h\to 0} \frac{\cancel{h}(x^2+2y+h)}{\cancel{h}} = \underline{x^2+2y}$$

16

Panel 17

How to Really find partial derivatives

f_x: keep y const, differentiate as usual for x

f_y: keep x const, diff as usual for y

chain, product, quotient

17

Panel 18

Ex: $f(x,y) = x^3 + x^2y^3 - 2y^2$. Find

$f_x(2,1)$: $f_x(x,y) = 3x^2 + 2xy^3$

$\Rightarrow f_x(2,1) = 3\cdot 2^2 + 2\cdot 2\cdot 1^3 = 12 + 4 = \underline{\underline{16}}$

$f_y(2,1)$: $f_y(x,y) = 3y^2x^2 - 4y$

$f_y(2,1) = 3\cdot 1^2\cdot 2^2 - 4\cdot 1 = \underline{\underline{8}}$

18

Panel 19

3D Example: $f(x,y,z) = \boxed{x}\, z\, e^{x^2+y^2}$. Find

$f_x(x,y,z) = z e^{x^2+y^2} + xz \cdot e^{x^2+y^2} \cdot 2x$

$f_y(x,y,z) = xz \cdot e^{x^2+y^2} \cdot 2y$

$f_z(x,y,z) = x e^{x^2+y^2}$

19