Panel 1

Summary

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}\,'(t) = \langle x', y', z' \rangle$

$\vec{T}(t) = r' / \|r'\|$

$\vec{N}(t) = T' / \|T'\|$

$\vec{B}(t) = T \times N$ ✓

$$\kappa(t) = \frac{\|T'\|}{\|r'\|} = \frac{\|r' \times r''\|}{\|r'\|^3}$$

Panel 2

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit normal and binormal vectors at $t=0$

$T(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$

$N(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

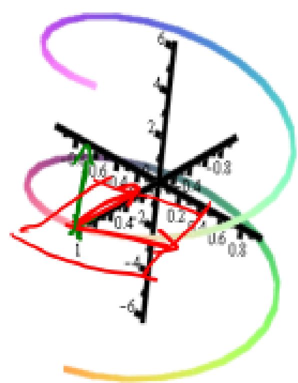
$B(t) = \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle$

$t=0$:

$\Rightarrow T = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$

$\Rightarrow N = \langle -1, 0, 0 \rangle$

$\Rightarrow B = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle$



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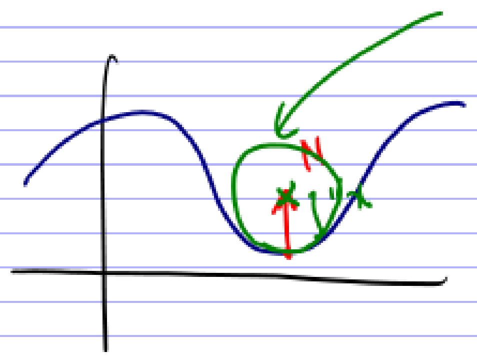
Panel 3

Def: The plane determined by T and N is called supporting plane or osculating plane.

Latin: osculum = kiss

Def: The circle in the osculating plane with radius $r = 1/\kappa$ is called the osculating circle



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Panel 4

Ex: Find the osculating plane of

$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ at $P(0, 1, \pi/2)$.

$t = \pi/2$

$T = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$

$N = \langle -\cos(t), -\sin(t), 0 \rangle$

$B = \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle$

Supporting plane contains $T = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle$

$N = \langle 0, -1, 0 \rangle$

$T \times N = B = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

Plane with $\vec{n} = \langle 1, 0, 1 \rangle$: $x + z + D = 0$

$x + z - \frac{\pi}{2} = 0$

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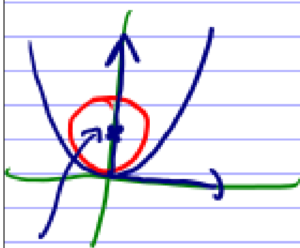
Panel 5

Ex: Find the osculating circle to $y = x^2$ at $(0,0)$.



$(0, \frac{1}{2})$

r'

r''

$$\kappa = \frac{|f''(x)|}{(1+(f')^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$$

κ at $(0,0)$ is $\kappa = 2$

radius is $\frac{1}{2} = \frac{1}{\kappa}$

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

$$\|\ \| = \begin{cases} \sqrt{x^2} = |x| & 1D \\ \sqrt{x^2+y^2} & 2D \\ \sqrt{x^2+y^2+z^2} & 3D \end{cases}$$

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Panel 6

Motion in Space

Suppose $\vec{r}(t)$ represents the motion or path of some object (particle) through space & time!

$r(t)$ = path or motion

$r'(t)$ = velocity   call $r'(t) = \vec{v}(t)$

$\|\vec{r}'\|$ = speed $\|r'\| = \|v\| = s(t)$

$r'' = (r')' = (v)'$ = acceleration $= \vec{a}(t)$

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Panel 7

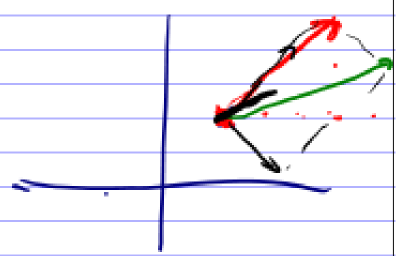
Ex: Suppose the path of a particle at time t is $\vec{r}(t) = \langle t^3, t^2 \rangle$. Find velocity, speed, and acceleration when $t=1$. Illustrate.

$\vec{v}(t) = \langle 3t^2, 2t \rangle \Rightarrow \vec{v}(1) = \langle 3, 2 \rangle$

$s(t) = \sqrt{9t^4 + 4t^2} \Rightarrow s(1) = \sqrt{13}$

$\vec{a}(t) = \langle 6t, 2 \rangle \Rightarrow \vec{a}(1) = \langle 6, 2 \rangle$

$r(t) = \langle t^3, t^2 \rangle \Rightarrow r(1) = \langle 1, 1 \rangle$



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Panel 8

Ex: A particle starts at $P(1,0,0)$ with initial velocity $\langle 1, -1, 1 \rangle$. The acceleration is $\vec{a}(t) = \langle 4t, 6t, 1 \rangle$. Find velocity, speed, and position.

$\vec{a}(t) = \langle 4t, 6t, 1 \rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 2t^2, 3t^2, t \rangle + C = \langle 2t^2, 3t^2, t \rangle + \langle 1, -1, 1 \rangle$

$\vec{v}(0) = \langle 1, -1, 1 \rangle = C$

$s(t) = \sqrt{(2t^2+1)^2 + (3t^2-1)^2 + (t+1)^2}$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \frac{2}{3}t^3, t^3, \frac{1}{2}t^2 \rangle + \langle t, -t, t \rangle + D$

$\vec{r}(0) = \langle 1, 0, 0 \rangle = D$

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Panel 9

Ex: An object ~~with mass m~~ moves in a circle with constant angular speed ω. Find the force acting on the object and illustrate.

$r(t) = \langle \cos(\omega t), \sin(\omega t) \rangle$

$v(t) = \langle -\omega \sin(\omega t), \omega \cos(\omega t) \rangle$, $\|v\| = s = \cancel{1}\,\omega$

$[\langle \cos(t), \sin(t) \rangle$ and $\langle \cos(5t), \sin(5t) \rangle]$

$F = m\vec{a} = m \langle -\omega^2 \cos(\omega t), -\omega^2 \sin(\omega t) \rangle =$
$-m\omega^2 r(t)$

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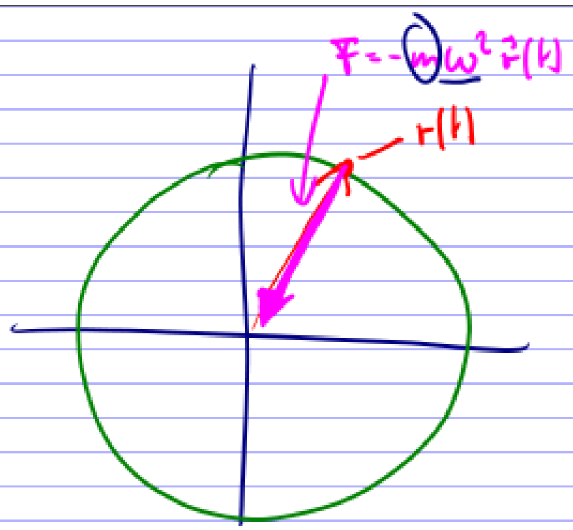
Panel 10



Need to push particle down in $-\vec{r}$ direction, harder if object moves faster and is heavier

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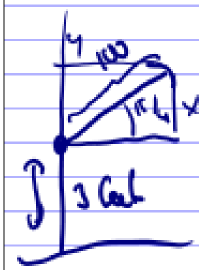
Panel 11

12:02

Application of Motion

A baseball is hit 3 feet above ground at 100 feet per second and at an angle of Pi/4 with respect to the ground. Find the maximum height reached by the baseball. Will it clear a 10-foot high fence located 300 feet from home base?

$\vec{a} = \langle 0, -g \rangle$



$\vec{v}(t) = \langle 1, -gt \rangle + C$ $\quad c_1 = \frac{100}{\sqrt{2}} - 1$ / $c_2 = \frac{100}{\sqrt{2}}$

$\vec{v}(0) = \langle \frac{100}{\sqrt{2}}, \frac{100}{\sqrt{2}} \rangle = \langle 1, 0 \rangle + \langle c_1, c_2 \rangle$

$\rightarrow \vec{v}(t) = \langle 1, -gt \rangle + \left(\frac{100}{\sqrt{2}} - 1, \frac{100}{\sqrt{2}} \right) = \langle \frac{100}{\sqrt{2}}, -gt + \frac{100}{\sqrt{2}} \rangle$

$\vec{r}(t) = \langle \frac{100}{\sqrt{2}} t + d_1, -\frac{1}{2} g t^2 + \frac{100}{\sqrt{2}} t + d_2 \rangle, \quad \vec{r}(0) = \langle d_1, d_2 \rangle = \langle 0, 3 \rangle$

$\vec{r}(t) = \langle \frac{100}{\sqrt{2}} t, -\frac{1}{2} g t^2 + \frac{100}{\sqrt{2}} t + 3 \rangle$ $\underline{\text{lect}}$ HW

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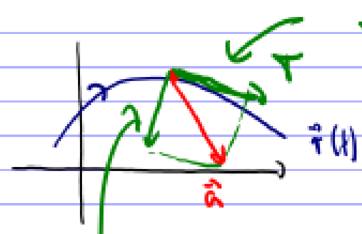
Panel 12

Tangential and Normal Components of Acceleration

tangential comp. of $\vec{a}$



normal comp. of acceleration

Acceleration (force) can be divided into two parts.

- one part to change speed : tangential comp. a_T
- one part to change direction : normal comp. a_N

$\vec{a} = a_T \vec{T} + a_N \vec{N}$

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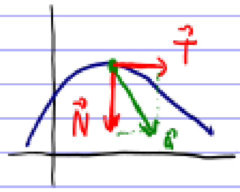
Panel 13



$$\vec{a} = a_t \cdot \vec{T} + a_N \vec{N}$$

$$T = r'/\|r'\| = \frac{v}{s} \Rightarrow v = s \cdot T$$

$$\frac{d}{dt}(v) = \vec{a} = s'T + sT'$$

Recall: $\vec{N} = T'/\|T'\| \Rightarrow sT' = N \cdot \|T'\| = N \cdot s\kappa$ $\qquad \kappa = \frac{\|T'\|}{\|r'\|} = \frac{\|T'\|}{s}$

$$\vec{a} = s'T + s^2 \kappa N$$

$$\Rightarrow \boxed{a_T = s' \qquad a_N = s^2 \kappa}$$

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Panel 14

$\vec{a} = a_t \vec{T} + a_N \vec{N}$ with

$a_T = s'$ and $a_N = s^2 \kappa$

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} \Rightarrow \underline{a_N} = s^2 \frac{\|v \times a\|}{s^3} = \underline{\frac{\|v \times a\|}{s}}$$

$$a = s'T + s^2 \kappa N \quad \Big| \cdot v = sT$$

$$a \cdot v = s'T \cdot sT + s^2 \kappa N \cdot sT =$$

$$= s's\, T \cdot T + s^3 \kappa \cancel{N \cdot T}^{0}$$

$$= s's$$

$$\Rightarrow \qquad s' = \frac{a \cdot v}{s}$$

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Panel 15

Theorem: $\vec{a} = a_T \vec{T} + a_N \vec{N}$ where

tang. component $a_T = \frac{v \cdot a}{s}$

normal component $a_N = \frac{\|v \times a\|}{s}$

Ex: $r(t) = \langle t^2, t^2, t^3 \rangle$ - find a_T and a_N at $t=1$

$v(t) = \langle 2t, 2t, 3t^2 \rangle$ $\quad v(1) = \langle 2,2,3 \rangle$

$s(1) = \sqrt{17}$

$a(t) = \langle 2, 2, 6t \rangle$ $\quad a(1) = \langle 2,2,6 \rangle$

$v \cdot a = 26$ $\quad \begin{vmatrix} i & j & k \\ 2 & 2 & 3 \\ 2 & 2 & 6 \end{vmatrix} = \langle 6, -6, 0 \rangle$ $\quad a_T = \frac{26}{\sqrt{17}}$ $\quad a_N = \frac{\sqrt{72}}{\sqrt{17}}$

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Panel 16

Quiz 4

Suppose $\vec{r}(t) = \langle t^2, 2, t \rangle$ is a vector-valued function (aka space curve), representing the position of a particle. Find the following:

1. The velocity at $P(0,0,0)$
2. The speed at $P(0,0,0)$
3. The acceleration at $P(0,0,0)$
4. The unit tangent $\vec{T}(t)$ at $P(0,0,0)$
5. Harder! The unit normal vector $\vec{N}(t)$ at $P(0,0,0)$
6. The bi-normal vector $\vec{B}(t)$ at $P(0,0,0)$
7. The curvature k at $P(0,0,0)$
8. The tangential component of the acceleration a_T at $P(0,0,0)$
9. The normal component of the acceleration a_N at $P(0,0,0)$
10. The osculating plane at $P(0,0,0)$
11. The osculating circle at $P(0,0,0)$

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