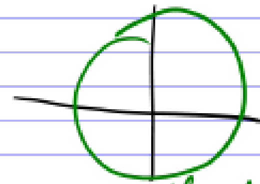
Panel 1

Last Time



$\vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ parametric "functions" or equations

Length a Curve: $\vec{r}(t)$ as t goes from $\underline{a}$ to $\underline{b}$ is

$$L = \int_a^b \|\vec{r}\,'(t)\|\,dt$$ in 2D, 3D, etc

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Panel 2

$r(t) = \langle 12t, 8t^{3/2}, 3t^2 \rangle$, $0 \le t \le 1$

$r'(t) = \langle 12, \underline{12t^{1/2}}, 6t \rangle$

$$\int_0^1 \|r'(t)\|\,dt = \int_0^1 \sqrt{144 + 144t + 36t^2}\,dt =$$

$$= \int_0^1 \sqrt{36(4+4t+t^2)}\,dt =$$

$$= \int_0^1 6\sqrt{(2+t)^2}\,dt = \int_0^1 6(2+t)\,dt = \blacksquare$$

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Panel 3

$\langle 3\sin(t), 4t, 3\cos(t)\rangle$ start at $(0,0,3)$ go for 5 m

$$s = L(t_0) = \int_0^{t_0} \sqrt{3^2\cos^2(t) + 16 + 3^2\sin^2(t)}\, dt =$$
$$= \int_0^{t_0} \sqrt{25}\, dt = \int_0^{t_0} 5\, dt = 5t_0 \Rightarrow t_0 = 1$$

Thus, are at $\langle 3\sin(1), 4, 3\cos(1)\rangle$

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Panel 4

Blast from the past: find length of $y = f(x)$, $x = t$ from a to b,

$$r(t) = \langle t, f(t)\rangle \qquad \Rightarrow f(x) = f(t)$$

$$L = \int_a^b \sqrt{1^2 + (f'(t))^2}\, dt = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int \sqrt{(x')^2 + (y')^2}\, dt$$

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Panel 5

It is possible for one curve to have many different parametrizations:

Ex: $r_1(t) = \langle t, t^2 \rangle$ $t = 0, 1$

$x = t,\ y = t^2 = x^2 \to y = x^2$

Same as: $r_2(t) = \langle t^3, t^6 \rangle$ $t = 0$ to 1

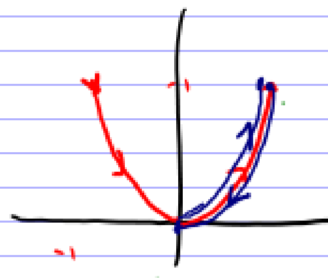
$x = t^3, y = t^6 \Rightarrow y = x^2$

Same as: $\langle \sin(t), \sin^2(t) \rangle$, $t = 0$ to $\frac{\pi}{2}$

Note that $\langle t, t^2 \rangle$, $t \in [-1, 1]$ and

$\langle t^2, t^4 \rangle$, $t \in [-1, 1]$ are

$y = x^2$ ✓ Different!



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Panel 6

Ex: $r_1(t) = \langle t, t^2 \rangle$ and $r_2(t) = \langle t^3, t^6 \rangle$, $t = 0 - 1$, describes the same curve $\underbrace{}_{y = x^2}$

Thm: The length of the curve does not depend on the parametrization.

$$du = 3t^2 dt$$
$$u = t^3$$

$$L_1 = \int_0^1 \sqrt{1 + 4t^2}\, dt$$

$$L_2 = \int_0^1 \sqrt{9t^4 + 36t^{10}}\, dt = \int_0^1 \sqrt{9t^4(1 + 4t^6)}\, dt = \int_0^1 3t^2 \sqrt{1 + 4t^6}\, dt$$

$$= \int_0^1 \sqrt{1 + 4u^2}\, du$$

Quiz on Wed!

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Panel 7

Def: A curve $r(t)$ is called smooth if $\vec{r}\,'(t) \neq 0$ $-(0,0,0)$

i.e. the components of r' are not simultaneously zero.

Def: If $r(t)$ is a smooth curve then

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$ is unit tangent vector!

Ex: $r(t) = \langle t, t^2 \rangle$ find $T(t)$. Is $r(t)$ smooth for all t?

$r(t) = \langle t, t^2 \rangle$ ← smooth

$r'(t) = \langle 1, 2t \rangle$, $\|r'(t)\| = \sqrt{1+4t^2}$

$$T(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$$

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Panel 8

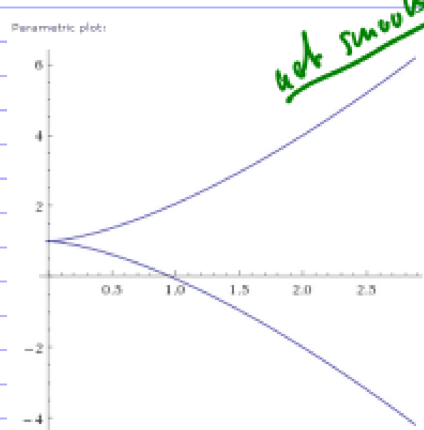
Smooth and Not Smooth?

smooth $r_1(t) = \langle \overbrace{t-1}^{x}, \overbrace{t^2-1}^{y} \rangle$ $r_1'(t) = \langle 1, ? \rangle$

$r_3(t) = \langle \cos(t), \sin(t) \rangle$ smooth (circle)

$r_2(t) = \langle 2t^2, 3t^3+1 \rangle$ $r_2'(t) = \langle 4t, 9t^2 \rangle$, $t=0$ not smooth

not smooth



$x = t-1$

$\Rightarrow t = x+1$

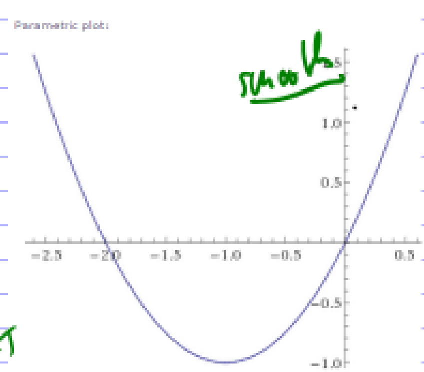
$y = t^2-1$

$y = (x+1)^2-1$

$= x^2+2x+1-1$

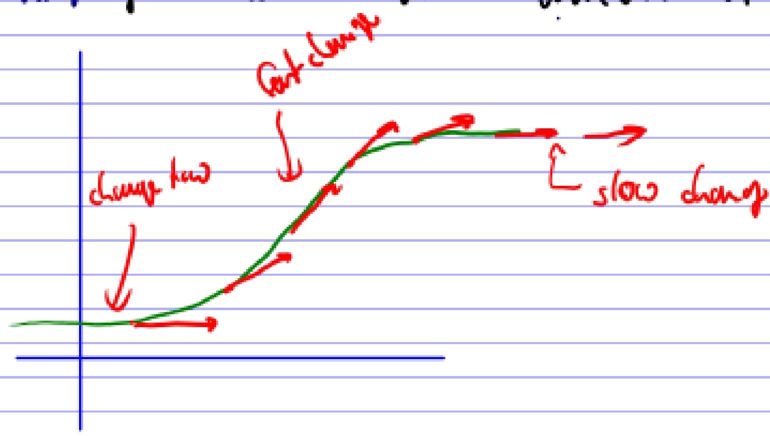
$= x(x+2)$

smooth



Panel 9

Unit Tangent gives the "normalized" direction of change.



Want: rate of change of unit tangent.

$$\overset{\text{kappa}}{\kappa} = \left\| \frac{dT}{dt} \right\| \quad \text{is curvature of } \vec{r}$$

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Panel 10

Def. Curvature of smooth curve $r(t)$ is

$$\kappa = \left\| \frac{dT}{dt} \right\| = \frac{\|T'(t)\|}{\|r'(t)\|}, \text{ where } T \text{ is unit tangent}$$

Ex. $r(t) = \langle 3t, 1+2t, t+1 \rangle$ Find curvature of $\vec{r}$:

$r'(t) = \langle 3, 2, 1 \rangle$

$\|r'\| = \sqrt{14}$

$\Big\} \; T(t) = \frac{1}{\sqrt{14}} \langle 3, 2, 1 \rangle$

$\Rightarrow \kappa = \left\| \frac{dT}{dt} \right\| = \|0\| = 0$ makes sense for straight line $\vec{r}$

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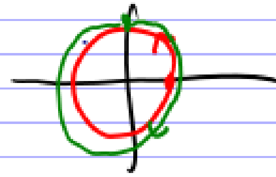
Panel 11

Ex: Curvature of circle, radius R, centered at $(0,0)$

$\vec{r}(t) = \cancel{\langle R\sin(t), R\cos(t)\rangle}$

$= \langle R\cos(t), R\sin(t)\rangle$ ← standard circle



Need r', T, T'

$r'(t) = \langle -R\sin(t), R\cos(t)\rangle$, $\|r'\| = \sqrt{R^2\sin^2(t) + R^2\cos^2(t)} = R$

$T(t) = \frac{1}{\cancel{R}} \langle -\cancel{R}\sin(t), \cancel{R}\cos(t)\rangle = \langle -\sin(t), \cos(t)\rangle$

$T'(t) = \langle -\cos(t), -\sin(t)\rangle$, $\|T'\| = 1$

$\kappa = \frac{\|T'\|}{\|r'\|} = 1/R$

circles have constant curvature; large circle ⇒ small curvature

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Panel 12

Ex: Find curvature of $\langle t, t^2\rangle$ $y = x^2$

$\kappa = \frac{\|T'\|}{\|r'\|}$

$r'(t) = \langle 1, 2t\rangle$, $\|r'\| = \sqrt{1+4t^2}$

$T(t) = \frac{r'}{\|r'\|} = \frac{1}{\sqrt{1+4t^2}}\langle 1, 2t\rangle = (1+4t^2)^{-1/2}\langle 1, 2t\rangle$

$T'(t) = -\cancel{\tfrac{1}{2}}(1+4t^2)^{-3/2} \cdot \cancel{8}^{\,4} t\langle 1, 2t\rangle + (1+4t^2)^{-1/2}\langle 0, 2\rangle$

$= (1+4t^2)^{-1/2}\left[\frac{-4t}{1+4t^2}\langle 1, 2t\rangle + \langle 0, 2\rangle\right] =$

$= (1+4t^2)^{-1/2}\left[\frac{-4t}{1+4t^2}, \frac{\cancel{-8t^2} + 2(1+\cancel{4t^2})}{1+4t^2}\right] = (1+4t^2)^{-1/2}\left[\frac{-4t}{1+4t^2}, \frac{2}{1+4t^2}\right]$

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Panel 13

$$T'(t) = (1+4t^2)^{-3/2}\langle -4t, 2\rangle$$

Note: T' is almost always bad news!

$$\|T'\| = (1+4t^2)^{-3/2}\sqrt{16t^2+4} =$$

$$= (1+4t^2)^{-3/2}\, 2\sqrt{4t^2+1} = \frac{2}{1+4t^2}$$

$$\kappa = \frac{\|T'\|}{\|r'\|} = \frac{2}{(1+4t^2)}\cdot\frac{1}{(1+4t^2)^{1/2}} = \frac{2}{(1+4t^2)^{3/2}}$$ is

curvature of a parabola (curvature is largest for $t=0$, i.e. at vertex)

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Panel 14

Ex: Curvature of $r(t) = \langle t, t^2, t^3\rangle$

$$r'(t) = \langle 1, 2t, 3t^2\rangle, \quad \|r'\| = \sqrt{1+4t^2+9t^4}$$

$$T(t) = (1+4t^2+9t^4)^{-1/2}\cdot\langle 1, 2t, 3t^2\rangle$$

$T'(t) =$ give up!

Give me a better formula, PLEASE!

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Panel 15

12:06

Thm: The curvature of $\vec{r}(t)$ is

$$\kappa = \frac{\|T'\|}{\|r'\|} = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

Ex: $\vec{r}(t) = \langle t, t^2 \rangle \sim \langle t, t^2, 0 \rangle$

$r'(t) = \langle 1, 2t, 0 \rangle$

$r''(t) = \langle 0, 2, 0 \rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$\Rightarrow \kappa = \frac{\|\langle 0,0,2 \rangle\|}{\left(\sqrt{1+4t^2}\right)^3} = \frac{2}{(1+4t^2)^{3/2}}$$

nice!

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Panel 16

Thm: $r(t)$ a space curve s.t. $\|\vec{r}(t)\| = 1$. Then $\vec{r}(t) \cdot \vec{r}'(t) = 0$, i.e. $\vec{r}$ and $\vec{r}'$ are perpendicular!!!

Look at $\|\vec{r}(t)\|^2 = 1$

$\vec{r} \cdot \vec{r} = 1 \quad \Big| \frac{d}{dt}$

$r' \cdot r + r r' = \frac{d}{dt}(1) = 0$

$2 r r' = 0$

$\Rightarrow \vec{r} \cdot \vec{r}' = 0$

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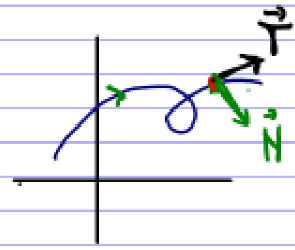
Panel 17

Def: If $\vec{T}(t)$ is the unit tangent to a space curve $\vec{r}(t)$ then $\vec{T}'(t)$ is perpendicular to $\vec{T}(t)$. We define:



$$\vec{N}(t) = \frac{T'(t)}{\|T'\|} \text{ is \underline{unit normal vector}}$$

$$\vec{B}(t) = \vec{T} \times \vec{N} \text{ is (unit) \underline{binormal vector}}$$

At every point $\vec{r}(t)$ the vectors $\vec{T}, \vec{N}, \vec{B}$ form a local coordinate system.

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Panel 18

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit normal and binormal vectors at $t = 0$

$$r'(t) = \langle -\sin(t), \cos(t), 1 \rangle \quad , \|r'\| = \sqrt{2}$$

$$\Rightarrow \vec{T} = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$$

$$\Rightarrow \quad \frac{dT}{dt} = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle \; , \left\| \frac{dT}{dt} \right\| = 1/\sqrt{2} \cdot 1$$

Fun fact: $\vec{T} \cdot \vec{N} = 0 \quad , \|\vec{N}\| = \left\| \frac{dT}{dt} \right\| = 1$

$$\vec{N} = \frac{T'}{\|T'\|} = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$\vec{B} = \begin{array}{ccc} i & j & k \end{array}$$
$$\vec{B} = \begin{vmatrix} -\frac{1}{\sqrt{2}}\sin & \frac{1}{\sqrt{2}}\cos(t) & \frac{1}{\sqrt{2}} \\ -\cos & -\sin & 0 \end{vmatrix} = \left\langle \frac{1}{\sqrt{2}}\sin(t), -\frac{1}{\sqrt{2}}\cos(t), \frac{1}{\sqrt{2}} \right\rangle$$

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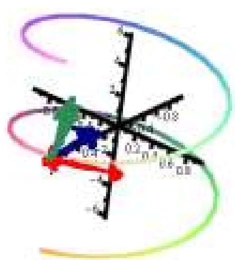
Panel 19

Ex: Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Find tangent, unit normal and binormal vectors at $t=0$

$$T = \frac{1}{\sqrt{2}} \langle 0,1,1 \rangle$$

$$N = \langle 1,0,0 \rangle$$

$$B = \frac{1}{\sqrt{2}} \langle 0,-1,1 \rangle$$



$$r(0) = \langle 1,0,0 \rangle$$

$$r(t) = \langle \cos(t), \sin(t), \cancel{t} \rangle$$

$$N = \langle -\cos(t), -\sin(t), 0 \rangle$$

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Panel 20

Summary

Exam I next Monday Feb 25

$\vec{r}(t) =$

$\vec{r}\,'(t) =$

$\vec{T}(t) =$

$\vec{N}(t) =$

$\vec{B}(t) =$

$\kappa(t) =$

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