Panel 1

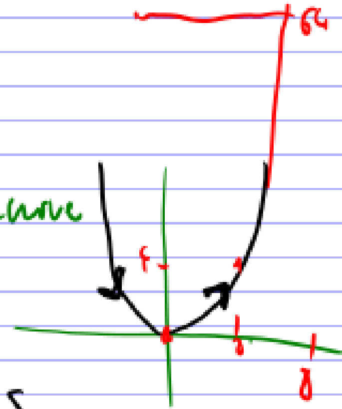
Last Time

parametric equation

$\vec r(t) = \langle x(t), y(t) \rangle$

space curve

$\vec r(t) = \langle x(t), y(t), z(t) \rangle$



Ex: $\langle t\cos(t), t, t\sin(t) \rangle$, $\langle t^3, t^6 \rangle$

x, $y = t^6 = (t^3)^2 = x^2$

limits of $\vec r(t)$ ✓

deriv. of $\vec r(t)$ ✓

integrals of $\vec r(t)$ ✓

$2^{1/3}$

$\langle t, t^2 \rangle \sim y = x^2 \quad [0,2]$

$2^{1/6}$

$\langle t^3, t^6 \rangle \sim y = x^2 \quad [0,2]$

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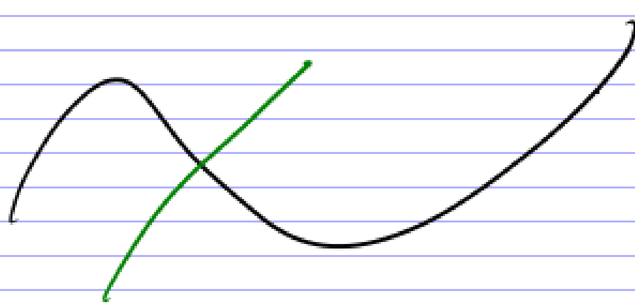
Panel 2

$$\int t \cdot \sin(\pi t)\,dt = -\frac{t}{\pi}\cos(\pi t) + \frac{1}{\pi}\int \cos(\pi t)\,dt =$$

$u = t \qquad u' = 1$

$v' = \sin(\pi t) \qquad v = -\frac{1}{\pi}\cos(\pi t)$

$$= -\frac{t}{\pi}\cos(\pi t) + \frac{1}{\pi^2}\sin(\pi t) + C$$



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Panel 3

$r_1(t) = \langle t, t^2, t^3 \rangle$, $r_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle$

$t = 1+2s \Rightarrow$

$t^2 = 1+6s \Rightarrow (1+2s)^2 = 1+6s$

$t^3 = 1+14s \qquad \cancel{1}+4s+4s^2 = \cancel{1}+6s$

$-2s+4s^2=0$

$-2s(1-2s)=0 \quad \begin{pmatrix} s=0 \\ t=1 \end{pmatrix} \begin{pmatrix} s=\frac{1}{2} \\ t=2 \end{pmatrix}$

They intersect twice:

r_1 at $t=1$ = r_2 at $s=0$

r_1 at $t=2$ = r_2 at $s=\frac{1}{2}$

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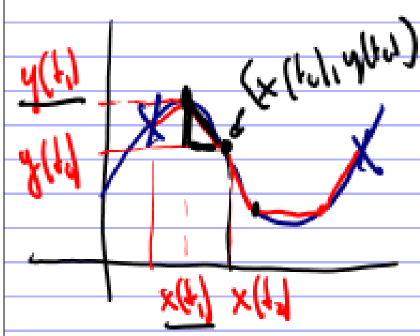
Panel 4

Length of a Curve $r(t) = \langle x(t), y(t) \rangle$



$d = \sqrt{\Delta x^2 + \Delta y^2}$

$d = \sqrt{\Delta t^2\left(\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2\right)} =$

$\Delta t \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$

$D \approx \lim_{\Delta t \to 0} \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t$ $\qquad \sqrt{a_1^2 + a_2^2} = \|\vec{a}\|$

$= \int_a^b \sqrt{(x')^2 + (y')^2}\, dt = \int_a^b \|r'(t)\|\, dt$

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Panel 5

Thm: Length of curve $r(t)$ is $C = \int_a^b \|r'(t)\|\,dt$

Ex: Length of $r(t) = \langle 2\cos(t), 2\sin(t)\rangle$ $\begin{cases} (2,0) \Rightarrow t=0 \\ (0,2) \Rightarrow t=\frac{\pi}{2} \end{cases}$

from $(2,0)$ to $(0,2)$

$$L = \int_0^{\pi/2} \|r'\|\,dt = \int_0^{\pi/2} \|\langle -2\sin(t), 2\cos(t)\rangle\|\,dt =$$

$$= \int_0^{\pi/2} \sqrt{4\sin^2(t) + 4\cos^2(t)}\,dt =$$

$$= \int_0^{\pi/2} 2\,dt = 2\cdot\left(\frac{\pi}{2} - 0\right) = \pi$$

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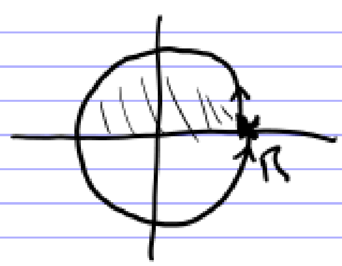
Panel 6

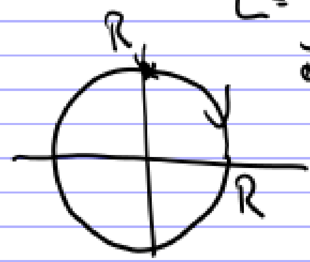
Show that circumference of a circle, radius R is $2\pi R$:

$r(t) = \langle R\cos(t), R\sin(t)\rangle \overset{?}{=} \langle R\sin(t), R\cos(t)\rangle$

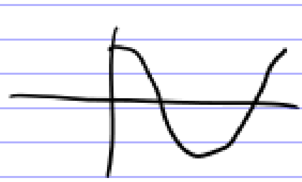
$t = 0$ to 2π

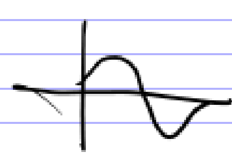
$$C = \int_0^{2\pi} \|r'(t)\|\,dt = \int_0^{2\pi} R\,dt = \underline{R\,2\pi}$$





$$\|r'\| = \sqrt{(-R\sin(t))^2 + (R\cos(t))^2} = \sqrt{R^2} = R$$





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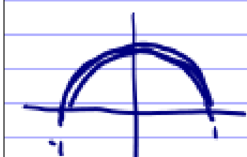
Panel 7

Ex: Compute length of $r(t) = \langle t, \sqrt{1-t^2} \rangle$, $t = -1$ to 1

$$L = \int_{-1}^{1} \|r'(t)\|\, dt = \int_{-1}^{1} \left\| \left\langle 1, \frac{-2t}{2\sqrt{1-t^2}} \right\rangle \right\| dt =$$

$$= \int_{-1}^{1} \sqrt{1 + \frac{t^2}{1-t^2}}\, dt =$$

$$= \int_{-1}^{1} \sqrt{\frac{1-t^2+t^2}{1-t^2}}\, dt = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, dt =$$

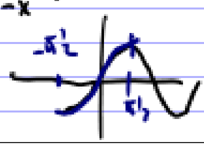


$y = \sqrt{1-x^2} \Leftrightarrow y^2 + x^2 = 1$

$$= \arcsin(t)\Big|_{-1}^{1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$

$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$



$\int_{\pi}^{0} \|r'\|\, dt$, $r' = \langle \cos(t), \sin(t) \rangle$, $t \in [\pi, 0]$

$\Rightarrow \pi$

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Panel 8

As far as length goes, different parametrizations of same curve give same length.

Thus: Pick easiest parametrization before working out length!!!

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