Panel 1

Last Time

Dot Product: $\vec{a}\cdot\vec{b} =$ ✓

Geometrically: $\dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \cos(\theta)$

✓ Projection of $\vec{b}$ onto $\vec{a}$: $\mathrm{proj}_{\vec{a}}(\vec{b}) = \dfrac{a\cdot b}{\|a\|^2}\vec{a}$

Length of $\mathrm{proj}_{\vec{a}}(\vec{b})$: $\mathrm{comp}_{\vec{a}}(\vec{b}) = \|\mathrm{proj}_{\vec{a}}(\vec{b})\| = \dfrac{a\cdot b}{\|a\|}$

Properties of dot product: $\vec{a}\cdot\vec{a} = \|a\|^2$

1

Panel 2

Cross Product: $a\times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2, \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix}$

Geometrically: $a\times b$ is a vector that is perp. to both $\vec{a}$ and $\vec{b}$, i.e. perp. to plane spanned by $\vec{a}$ and $\vec{b}$

$\|a\times b\| = |\sin\theta|\,\|a\|\,\|b\|$

= area of parallelogram $\vec{a}$ $\vec{b}$

Properties of cross product: $\vec{a}\times\vec{a} = 0$ ✓

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Panel 3

Dot Product: $a\cdot b$ is #

Cross Product: $a\times b$ is vector

Cross out the expressions that do not make sense. For the rest, is the answer a vector or a scalar?

| | |
|---|---|
| $a\cdot(b\times c)$ ✓ | $(a\cdot b)\times c$ ✗ |
| $(a\cdot b)\cdot c$ ✗ | $(a\times b)+c$ ✓ |
| $a\times(b\cdot c)$ ✗ | $(a\cdot b)\times(c\cdot d)$ ✗ |
| $\|a\|(b\cdot c)$ ✓ | $(a\cdot b)+c$ ✗ |
| $a\times(b\times c)$ ✓ | $(a\times b)\cdot(c\times d)$ ✓ |
| $a\cdot(b+c)$ ✓ | $\|a\|(b\times c)$ ✓ $\;|a\cdot b|$ ✓ |
| $(a\times b)+c$ ✓ $\;|a\times b|$ | $\|a\times b\|$ ✓ $\;\|a\cdot b\|$ ✗ |

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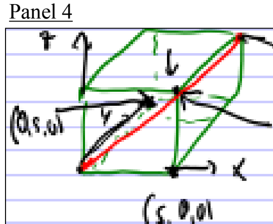
Panel 4




cube, side s

angle between $(s,s,s)$ and $(0,s,0)$

$$\frac{(s,s,s)\cdot(0,s,0)}{\sqrt{3s^2}\;\sqrt{s^2}} = \frac{s^2}{\sqrt{3}\cdot s\cdot s} = \frac{1}{\sqrt{3}} = \cos(\theta)$$

$\theta = 54.7°$

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Panel 5

## Quiz #2

Name: ______________

① Find the dot product $\langle 3,-2,1\rangle \cdot \langle 1,2,2\rangle$

② Which vector is perpendicular to $\langle 3,-2,1\rangle$, which one is parallel?

a) $\langle -6,4,-2\rangle$ b) $\langle 2,4,2\rangle$

Panel 6

③ Find the projection of $\langle 3,-1,-2\rangle$ onto $\langle 3,3,1\rangle$

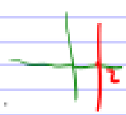
④ Find the cross product $\langle 3,-2,1\rangle \times \langle 2,-1,1\rangle$

Panel 7

## Of Lines and Planes

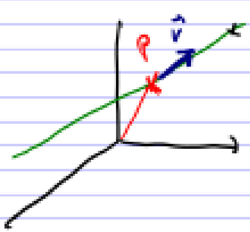
Line: $y = mx + b$ not that great, because

a) vertical line: $x = \#$




b) does not generalize, i.e

$z = m_1 x + m_2 y + b$ is not a line!




line has point P, dir $\vec{v}$.

then any point on line can be written as

$P + t\vec{v}$.

Panel 8

Def: If $P(x_0, y_0, z_0)$ is a point on a line, and $\vec{v} = \langle a,b,c\rangle$ is the direction of the line, then the (parametric) equation of the line is:

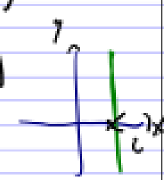
$$\ell(t) = P + t\vec{v}$$

$$= \langle x_0, y_0, z_0\rangle + t\langle a,b,c\rangle =$$

$$\langle x_0 + ta,\ y_0 + tb,\ z_0 + tc\rangle$$

Ex: vertical line through $x = 2$ (in $\mathbb{R}^2$)

$\ell(t) = \langle 2,0\rangle + t\langle 0,1\rangle =$



Panel 9

Ex: Find equation of a line

a) through $(5,1,3)$ and parallel to $\vec{v} = \langle 1,4,-2 \rangle$

$$\ell(t) = (5,1,3) + t\langle 1,4,-2 \rangle = \langle 5+t, 1+4t, 3-2t \rangle$$

b) through $P(1,2,3)$ and $Q(4,1,1)$ $\qquad \vec{PQ} = \langle 4-1, 1-2, 1-3 \rangle$

$$\ell(t) = (1,2,3) + t\langle -3,-1,-2 \rangle$$

Panel 10

Ex: At what point does $\langle 2,4,-3 \rangle + t\langle 1,-5,4 \rangle$ intersect the $xy$-plane

In $(xy)$-plane I know that $z = 0$

$$\ell(t) = \langle 2+t, 4-5t, -3+4t \rangle$$

$$-3+4t = 0$$
$$4t = 3$$
$$t = \tfrac{3}{4}$$

$$\ell\left(\tfrac{3}{4}\right) = \left\langle 2+\tfrac{3}{4}, 4-\tfrac{15}{4}, 0 \right\rangle$$

Panel 11

Ex: Suppose 2 lines are $\ell_1(t) = \langle 1+t, -2+3t, 4-t \rangle$ $\quad t\in \mathbb{R}$

$\ell_2(s) = \langle 2s, 3+s, -3+4s \rangle = \langle 0+2s, 3+s, -3+4s \rangle = \langle 0,3,-3 \rangle + \langle 2s, s, 4s \rangle + s\langle 2,1,4 \rangle$

a) Are the lines parallel? No

dir of $\ell_1$: $\langle 1,3,-1 \rangle$

dir of $\ell_2$: $\langle 2,1,4 \rangle$

b) Do they intersect in $\mathbb{R}^3$? $\qquad \ell(t) = \langle t, 1, 2+3t \rangle$

If they intersected, they would have a point in common

$$\ell_1(t) = \ell_2(s) \iff \langle 1+t, -2+3t, 4-t \rangle = \langle 2s, 3+s, -3+4s \rangle$$

$1+t = 2s$, $\quad -2+3t = 3+s$, $\quad 4-t = -3+4s$

$t = 2s-1$, $\quad -2+3(2s-1) = 3+s \Rightarrow -2+6s-3$

1:20

Panel 12

$1+t = 2s$, $\quad -2+3t = 3+s$, $\quad 4-t = -3+4s$

$\Rightarrow t = 2s-1$

Don't intersect!

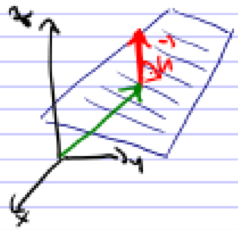
$$-2+3(2s-1) = 3+s$$
$$-2+6s-3 = 3+s$$
$$5s = 8, \quad s = \tfrac{8}{5} \quad \Rightarrow t = \tfrac{16}{5} - 1 = \tfrac{11}{5}$$

$$\Rightarrow 4 - \tfrac{11}{5} \overset{?}{=} -3 + 4\cdot\tfrac{8}{5}$$

$$\overset{?}{\cancel{=}}\ \tfrac{42}{5}$$

Panel 13

Planes in $\mathbb{R}^3$



A plane in $\mathbb{R}^3$ is uniquely determined by

- 3 points, or
- 2 vectors + 1 point
- 1 vector perp. to plane + 1 point

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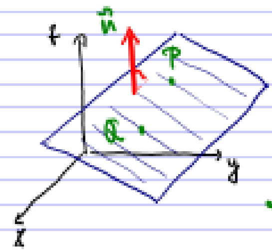
Panel 14

Suppose a plane goes through $P(x_0, y_0, z_0)$ and has normal vector $\vec{n} = \langle a, b, c \rangle$ (perp to plane)



Take any $Q(x, y, z)$ in plane.

Then $\vec{PQ} \cdot \vec{n} = 0$ ($\vec{PQ}$ perp. to $\vec{n}$)

$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle =$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{d \text{ const}}$ $\Rightarrow ax + by + cz = d$

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Panel 15

Def: The equation of a plane with normal vector $\vec{n} = \langle a, b, c \rangle$ through the point $P_0(x_0, y_0, z_0)$ is:

$ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$

Recall: $y = mx + b$ line $\quad Q(t) = P_0 + t\vec{v}$ always in any dimension

$ax + by + cz = d \Rightarrow cz = -ax - by + d$

$z = \# x + \# y + \#$ plane!

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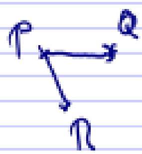
Panel 16

Scalar equation of plane through $P(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or

$ax + by + cz = d$

Ex: Plane through $P(1, 3, 2)$, $Q(3, -1, 6)$ and $R(5, 2, 0)$

Recall: $\vec{n} = \langle a, b, c \rangle$ is normal vector, i.e. perp. to plane. $\vec{n} = \vec{PQ} \times \vec{PR} = \langle 2, -4, 4 \rangle \times \langle 4, -1, -2 \rangle$



$\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 8 + 4, +18, 14 \rangle = \langle 12, 18, 14 \rangle$

$= 2\underline{\langle 6, 9, 7 \rangle}$

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Panel 17

$\vec{n} = \langle 6, 9, 7 \rangle \to 6x + 9y + 7z = d$

$R(5,2,0)$ is on the plane $\Rightarrow$

$6 \cdot 5 + 9 \cdot 2 + 7 \cdot 0 = d = 48$

plane: $6x + 9y + 7z = 48$

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Panel 18

Scalar equation of Plane through $P(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Ex: Angle between planes $x + y + z = 1$ and $x - 2y + 3z = 1$

normal to $1^{st}$ plane: $\langle 1, 1, 1 \rangle$

$2^{nd}$ plane: $\langle 1, -2, 3 \rangle$

$$\cos(\theta) = \frac{\langle 1,1,1 \rangle \cdot \langle 1, -2, -3 \rangle}{\sqrt{3} \ \sqrt{14}}$$

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