Panel 1

Last Time: Dot Product

$$\vec{V} \cdot \vec{W} = \langle V_1, V_2, V_3 \rangle \cdot \langle W_1, W_2, W_3 \rangle$$

$$= V_1 W_1 + V_2 W_2 + V_3 W_3$$

Properties

(1) $\vec{V} \cdot \vec{W}$ is a number (scalar) not a vector

(2) $\vec{V} \cdot \vec{V} = \|\vec{V}\|^2$! $\rightarrow$ V, W are orthog. !

(3) $\cos(\theta) = \dfrac{\vec{V} \cdot \vec{W}}{\|\vec{V}\| \, \|\vec{W}\|}$! iff $V \cdot W = 0$!

1

Panel 2

We also talked about "Directional angles" of $\vec{V}$

a) with x-axis $(\vec{i} = \langle 1, 0, 0 \rangle)$: $\cos(\theta_x) = V_1 / \|V\|$

b) with y-axis $(\vec{j} = \langle 0, 1, 0 \rangle)$: $\cos(\theta_y) = V_2 / \|V\|$

c) with z-axis $(\vec{k} = \langle 0, 0, 1 \rangle)$: $\cos(\theta_z) = V_3 / \|V\|$

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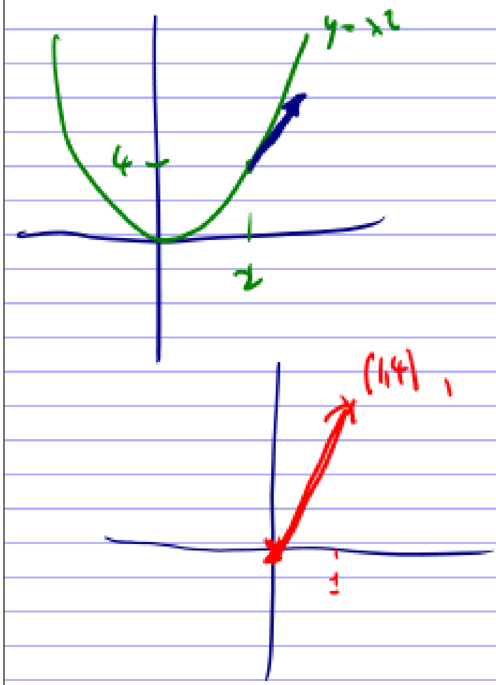
Panel 3



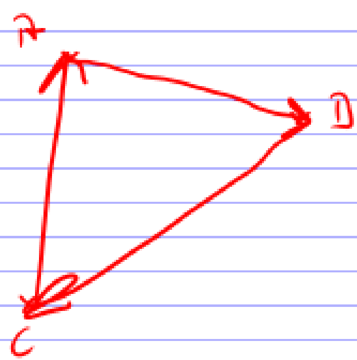
$y' = 2x$ at 2 is 4

$\Rightarrow$ slope of line is 4!

$\vec{w} = \langle 1,4 \rangle$

3

Panel 4

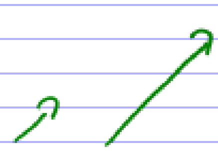


$$\| \vec{r} - \vec{r}_0 \| = \| \langle x - x_0, y - y_0, z - z_0 \rangle \|$$

$$\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = 1$$

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = 1$$

$$\frac{(4,0,4)\cdot(2,-1,0)}{\|\quad\|\cdot\|\quad\|} = \frac{8}{\sqrt{32}\cdot\sqrt{5}} = \cos(\theta)$$



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Panel 5

2 vectors are perp: if $\vec{v}\cdot\vec{w}=0$

parallel: if $\vec{v}=k\vec{w}$, $k\in\mathbb{R}$

$\langle -3, 9, 6\rangle$, $\langle 4, -12, -8\rangle$

$k\langle -3, 9, 6\rangle = \langle 4, -12, -8\rangle$? are parallel!

$-3k=4 \Rightarrow k=-\frac{4}{3}$

$\left(-\frac{4}{3}\right)\cdot 9 = -12$

$\left(-\frac{4}{3}\right)\cdot 6 = -8$

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Panel 6

$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})=0$ are orthogonal

$\vec{u}\cdot\vec{u}-\cancel{\vec{u}\vec{v}}+\cancel{\vec{v}\cdot\vec{u}}-\vec{v}\vec{v}=0$

$\|u\|^2-\|v\|^2=0$

$\|u\|=\|v\|$ i.e. same length!

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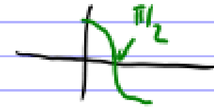
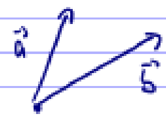
Panel 7

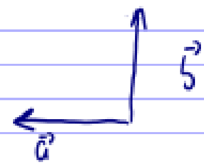
Picture Problems

$\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|} = \cos\theta \Leftrightarrow \vec{a}\cdot\vec{b} =$

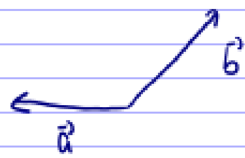




$\vec{a}\cdot\vec{b}$ { positive / zero / negative



$\vec{a}\cdot\vec{b}$ { positive / zero / negative



$\vec{a}\cdot\vec{b}$ { positive / zero / negative

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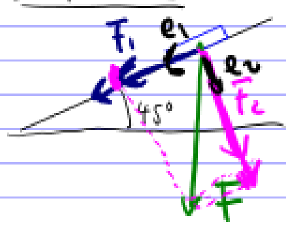
Panel 8

Application: Suppose a 10Kg block is on a 45° incline. What is the force pulling the block in the direction of the incline?



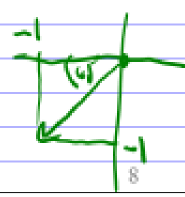
$F = \langle 0, -10\rangle$

Know: $F = F_1 + F_2$, F_1, F_2 are perp.

$F = k_1 e_1 + k_2 e_2$, e_1, e_2 are perp. unit vectors in dir. of F_1, F_2

$F = k_1 e_1 + k_2 e_2 \mid \cdot \vec{e}_1$ Need $e_1 = \frac{1}{\sqrt{2}}(-1,-1)$

$e_1 F = k_1 \vec{e}_1\cdot\vec{e}_1 + k_2 \vec{e}_2\cdot\vec{e}_1$
$= k_1$



$k_1 = \langle 0, -10\rangle \cdot \frac{1}{\sqrt{2}}(-1,-1)$
$= 10/\sqrt{2}$

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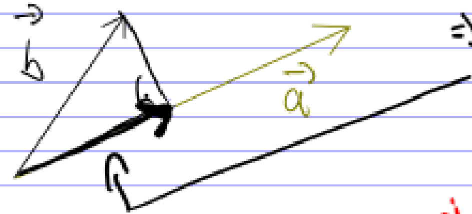
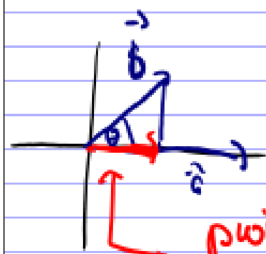
Panel 9

General Question: take two vectors $\vec{a}$ and $\vec{b}$. How much of $\vec{b}$ goes in the direction of $\vec{a}$?



$$\Rightarrow \text{proj}_{\vec{a}}(\vec{b}) = \underbrace{\underbrace{\frac{a\cdot b}{\|a\|}}_{\text{length}} \cdot \underbrace{\frac{\vec{a}}{\|a\|}}_{\text{dir}}}_{\text{vector}}$$

$$\cos(\theta) = \frac{\text{proj}}{\|b\|}$$



$$\text{proj}_{\vec{a}}(\vec{b}) = \|b\| \cdot \cos(\theta) = \cancel{\|b\|} \cdot \frac{a\cdot b}{\|a\|\cancel{\|b\|}}$$

is length of proj

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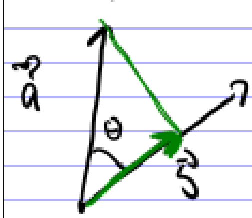
Panel 10

Projection Formula: $\text{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\,\vec{a}$

is proj. vector. It has length $\dfrac{\vec{a}\cdot\vec{b}}{\|a\|}$

Ex: Length and dir. of projection of $\vec{a}\ \langle 1,1,2\rangle$ onto $\vec{b} = \langle -2,3,1\rangle$: $\text{proj}_{\vec{b}}(\vec{a}) = \dfrac{\vec{a}\cdot\vec{b}}{\|b\|^2}\cdot\vec{b} =$



$$= \frac{3}{(\sqrt{14})^2}\cdot\langle -2,3,1\rangle.$$

$$= \frac{3}{14}\langle -2,3,1\rangle$$

Note: $\|\text{proj}_{b}(\vec{a})\| = \dfrac{3}{\sqrt{14}}$

Note: $\vec{a}\cdot\vec{b} = \langle 1,1,2\rangle\cdot\langle -2,3,1\rangle = -2+3+2 = 3 > 0 \rightarrow \theta < 90$

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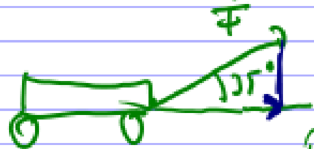
Panel 11

Application: A wagon is pulled a distance of $\underbrace{100\,m}$ by a constant force of 70 N, applied to a handle held at 35°. Find work done by F.



$\|\vec F\| = 70$ (in the dir. of dist)

Recall: Work = force · distance

Need: $\|proj_{\vec i}(\vec F)\| = \frac{\vec F\cdot\vec i}{\|\vec i\|} = \vec F\cdot\vec i = \cos(\theta)\,\|F\|\cdot\|i\|$.

$= \cos(35)\cdot 70 =$

$proj_{\vec b}(\vec a) = \frac{a\cdot b}{\|b\|}\vec b$

$= 57.34$

$\Rightarrow$ Work $= 100 \cdot 57.34 = \underline{5734}$

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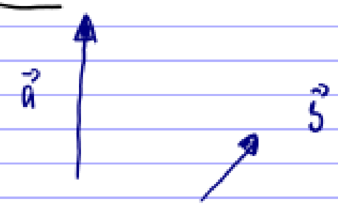
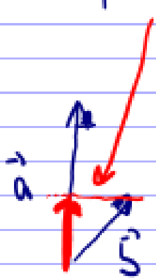
Panel 12

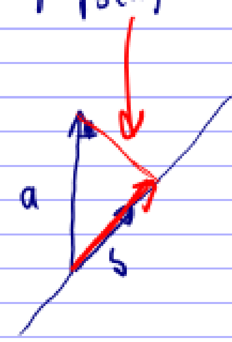
More picture Problems



find $proj_{\vec a}(\vec b)$

$proj_{\vec b}(\vec a)$





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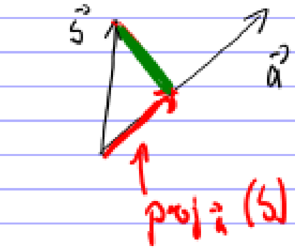
Panel 13

It looks like green vector is perp to $\vec{a}$



Prove it!

$proj_{\vec{a}}(\vec{b}) - \vec{b}$ is perp to $\vec{a}$

$\Rightarrow \left(proj_{\vec{a}}(\vec{b}) - \vec{b}\right) \cdot \vec{a} = ?\ 0$

HW

12:03

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Panel 14

So: Add / Subtract vector → nice, because get vector

Dot product of vectors → helpful, but strange (not vector)

Cross Product If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_3 b_2 - a_2 b_3 \rangle$$

Good news: $\vec{a} \times \vec{b}$ is vector!

Bad news: crazy! only works in $\mathbb{R}^3$

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Panel 15

How to memorize the cross product:

$$\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2,\ (a_1b_3 - a_3b_1),\ a_1b_2 - a_2b_1$$

Ex: $\langle 1,3,4 \rangle \times \langle 2,7,-5 \rangle = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} =$

$$\langle -15-28,\ -(-5-8),\ 7-6 \rangle$$

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Panel 16

$\langle 1,0,-2 \rangle \times \langle 0,2,-3 \rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 2 & -3 \end{vmatrix} \quad 0(-3)-(-2)(2),\ -\left((1)(-3)-(0)(-2)\right),\ (1)(2)-(0)(0)$$

$$0 - -4 \qquad -(-3+0) \qquad 2-0$$

$$\langle 4,\ 3,\ 2 \rangle$$

$\langle 1,0,0 \rangle \times \langle 0,1,0 \rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0,0,1 \rangle . \qquad \vec{i}\times\vec{j} = \vec{k}$$

$\langle 1,0,-2 \rangle \cdot \langle 0,2,-3 \rangle = 6$

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Panel 17

Properties: (1) $\vec{a} \times \vec{a} = 0$

(2) $\vec{a} \times \vec{b}$ is perp. to both $\vec{a}$ and $\vec{b}$

Proof $\vec{a} = \langle a_1, a_2, a_3 \rangle$. Compute $\vec{a} \times \vec{a}$

compute $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$ HW ☺

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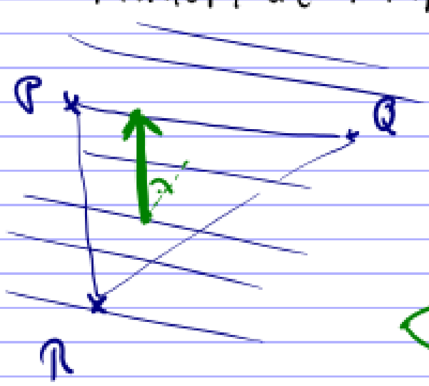
Panel 18

Ex: Find vector perpendicular to the plane through $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$



$\vec{PQ} \times \vec{PR} =$

$\langle -2-1, 5-4, -1-6 \rangle \times \langle 1-1, -1-4, 1-6 \rangle =$

$\langle -3, 1, -7 \rangle \times \langle 0, -5, -5 \rangle =$

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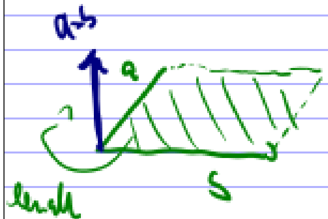
Panel 19

Geometric Int. of Cross-product

(a) $a \times b$ is perp to $\vec{a}$ and to $\vec{b}$!!!!

(b) $\|a \times b\| = \|a\| \, \|b\| \cdot \sin(\theta)$

(c) $\|a \times b\|$ is area of parallelogram determined by $\vec{a}$ and $\vec{b}$



Note: $a \times b = 0$ then $\vec{a}$, $\vec{b}$ are parallel.