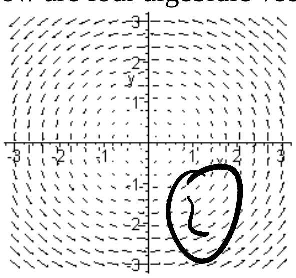
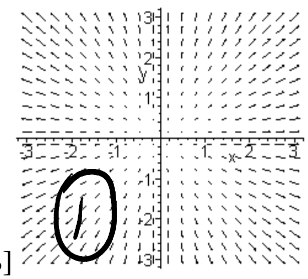
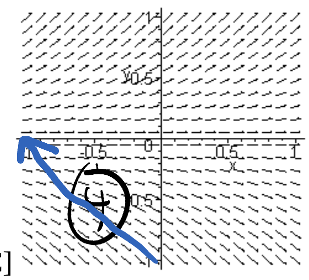
Math 2511: Calc III - Practice Exam 3

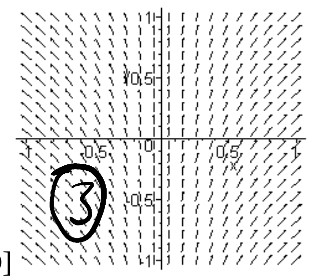
1. State the meaning or definitions of the following terms:
   a) vector field, conservative vector field, potential function of a vector field, volume, length of a curve, work, surface area, flux integral
   b) curl and divergence of a vector field F, gradient of a function
   c) $\iint_R dA$ or $\iint_R f(x,y)dA$ or $\iiint_Q f(x,y,z)dV$
   d) $\iint_R dS$ or $\int_C ds$ or $\int_C f(x,y)ds$ or $\int_C f(x,y)dx$ or $\int_C f(x,y)dy$ or $\iint_S g(x,y,z)dS$
   e) $\int_C \vec{F}\cdot d\vec{r}$ or $\iint_S \vec{F}\cdot\vec{n}dS$
   f) $\int_C M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$
   g) What does it mean when a "line integral is independent of the path"?
   h) State the Fundamental Theorem of Line Integrals. Make sure to know when it applies, and when it helps.
   i) State Green's Theorem. Make sure to know when it applies, and in what situation it helps.
   j) State Gauss' Theorem. Make sure to know when it applies, and in what situation it helps.

2. Below are four algebraic vector fields and four sketches of vector fields. Match them.

[A] 

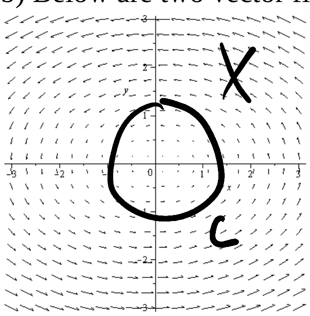
[B] 

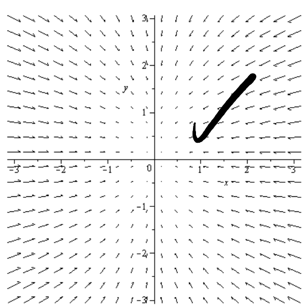
[C] 

[D] 

(1) $F(x,y) = \langle x, y\rangle$, (2) $F(x,y) = \langle -y, x\rangle$, (3) $F(x,y) = \langle x, 1\rangle$, (4) $F(x,y) = \langle 1, y\rangle$

b) Below are two vector fields. Which one is clearly not conservative, and why?





$\oint F\,dr \neq 0$

c) Say in the vector field [C] above you integrate over a straight line from (0,-1) to (-1,0). Is the integral positive, negative, or zero? negative

3. Are the following statements true or false:
   a) If the divergence of a vector is zero, the vector field is conservative. F
   b) If $F(x,y,z)$ is a conservative vector field then $curl(F) = 0$ T
   c) If a line integral is independent of the path, then $\int_C F\cdot dr = 0$ for every path C F
   d) If a vector field is conservative then $\int_C F\cdot dr = 0$ for every closed path C T

e) $\iint_R dA$ denotes the surface area of the region R F (area)

f) $\iint_R dS$ denotes the volume of the region R F (surface)

g) Can you apply the Fundamental Theorem of line integrals for the function $f(x,y,z) = xy\sin(z)\cos(x^2+y^2)$? F

h) Can you apply the Fundamental Theorem of line integrals for the vector field F
$F(x,y) =< 6xy^2 - 3x^2, 6x^2y + 3y^2 - 7 >$?

i) Can you apply Green's theorem for a curve C, which is a straight line from (0,0,0) to (1,2,3)? F

j) Can you apply the Divergence theorem to the plane *x+y+z=1* over [-1, 1] x [-1, 1]? F

4. Suppose that $F(x,y,z) =< x^3y^2z, x^2z, x^2y >$ is some vector field.
   a) Find div(F)
   $3x^2y^2z$

   b) Find curl(F)
   $(x^3y^2 - 2xy)\bar{e}_y + (2xz - 2x^3yz)\bar{e}_z$

   c) Find curl(curl(F))
   $-2x^3z\bar{e}_x + (-2z + 6x^2yz)\bar{e}_y + (3x^2y^2 - 2y)\bar{e}_z$

   d) Find div(curl(F))
   0

   e) grad., div., and curl of the vector field if appropriate for $< x^2, y^2, z^2 >$
   grad =n/a, div = 2x+2y+2z, curl = 0

   f) grad., div., and curl of the vector field if appropriate for $< \cos(y) + y\cos(x), \sin(x) - x\sin(y), xyz >$
   grad = n/a, div = $-y\sin(x) - x\cos(y) + xy$, curl = $(xz)\bar{e}_x - yz\bar{e}_y$

   g) grad., div., and curl of the vector field if appropriate for $f(x,y,z) = z\ln(x^2+y^2)$
   grad = $\frac{2zx}{x^2+y^2}\bar{e}_x + \frac{2zy}{x^2+y^2}\bar{e}_y + (\ln(x^2+y^2))\bar{e}_z$

5. Decide which of the following vector fields are conservative. If a vector is conservative, find its potential function
   a) $F(x,y) =< 2xy, x^2 >$
   ~~Not~~ conservative , $f(x,y) = x^2y + C$

   b) $F(x,y) =< e^x\cos(y), e^x\sin(y) >$
   Not conservative

   c) $F(x,y,z) =< \sin(y), -x\cos y, 1 >$
   Not conservative

   d) $F(x,y,z) =< 2xy, x^2 + z^2, 2zy >$
   ~~Not~~ conservative

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2+z^2 & 2zy \end{vmatrix} = <2z-2z, 0-0, 2x-2x>$$

$f(x,y,z) = \underline{\underline{x^2y + z^2y + C}}$

   e) $F(x,y) =< 6xy^2 - 3x^2, 6x^2y + 3y^2 - 7 >$

$f = 3x^2 y^2 - x^3 + y^3 - 7x + C$

f) $F(x, y) = \langle -2y^3 \sin(2x), 3y^2(1+\cos(2x)) \rangle$

g) $F(x, y) = \langle 4xy + z, 2x^2 + 6y, 2z \rangle$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 4xy+z & 2x^2+6y & 2z \end{vmatrix} = \langle 0-0, 4x+1$$

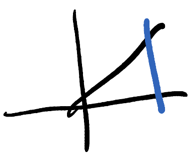
Not conservative

h) $F(x, y) = \langle 4xy + z^2, 2x^2 + 6yz, 2xz \rangle$

Not conservative

6. Evaluate the following integrals:

a) $\iint_R \cos(x^2)\,dA$ where R is the triangular region bounded by y = 0, y = x, and x = 1



$$\int_0^1 \int_0^x \cos(x^2)\,dy\,dx = \frac{1}{2}\sin(1)$$

b) $\iint_R dS$, where S is the portion of the hemisphere $f(x, y) = \sqrt{25 - x^2 - y^2}$ that lies above the circle $x^2 + y^2 \le 9$

$$\iint_C \sqrt{1+f_x^2+f_y^2}\,dA = \iint \frac{5}{\sqrt{25-x^2-y^2}}\,dA = \int_0^{2\pi}\int_0^3 \frac{5}{\sqrt{25-r^2}}\,r\,dr\,d\theta = 10\pi$$

c) $\int_C x^2 - y + 3z\,ds$ where C is a line segment given by $r(t) = \langle t, 2t, 3t \rangle$, $0 \le t \le 1$

$$\int_0^1 (t^2 - 2t + 9t)\sqrt{1+4+9}\,dt = \sqrt{14}\cdot 23/6$$

d) $\int_C F \cdot dr$ where $F(x, y) = \langle y, x^2 \rangle$ and C is the curve given by $r(t) = \langle 4 - t, 4t - t^2 \rangle$, $0 \le t \le 3$

$$\int_C y\,dx + x^2\,dy = \int_0^3 (4t - t^2)(-1)\,dt + (4-t)^2(4-2t)\,dt = \frac{15}{2}$$

$$\int_0^3 (-4t + t^2 + (4 - t^2)(4 - 2t))\,dt = \frac{15}{2}$$

e) $\int_C y\,dx + x^2\,dy$ where C is a parabolic arc given by $r(t) = \langle t, 1 - t^2 \rangle$, $-1 \le t \le 1$

$$\int_{-1}^1 (1-t^2)\,dt + t^2(-2t)\,dt = 4/3$$

f) $\iint_S (x + z)\,dS$ where S is the first-octant portion of the cylinder $y^2 + z^2 = 9$ between x = 0 and x = 4

can't do, need surface as $z = f(x,y)$

g) Find the flux of the vector field $F(x, y, z) = \langle x, y, z \rangle$ through the surface given by potion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane. Note that this surface is *not* closed.

$$\iint_R -f_x M - f_y N + P\, dA = \iint_{\text{circle}} 2x^2+2y^2+4-x^2-y^2\, dA = \int_0^{2\pi}\int_0^2 (4+r^2)\, r\, dr\, d\theta = \underline{24\pi}$$

7. For the following line integrals there is a short-cut you can use to simplify your computations (but justify your shortcut by quoting the appropriate theorem)

a) $\int_C F \cdot dr$ where $F(x,y,z) = < e^x \cos(y), -e^x \sin(y) >$ and C is the curve $r(t) = < 2\cos(t), 2\sin(t) >$, $0 \le t \le 2\pi$

F conservative, C closed curve $\Rightarrow \int F\, dr = 0$

b) $\int_C 2xyz\,dx + x^2 z\,dy + x^2 y\,dz$ where C is some smooth curve from (0,0,0) to (1,4,3)

$= f(1,4,3) - f(0,0,0) = 12$, where $f(x,y,z) = x^2 y z$ is potential function

c) $\int_C F \cdot dr$ where $F(x,y) = < y^3 + 1, 3xy^2 + 1 >$ and C is the upper half of the unit circle, from (1,0) to (-1,0).

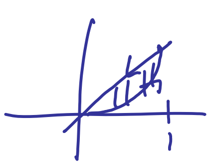
$= f(-1,0) - f(1,0) = -2$, where $f(x,y) = xy^3 + x$ is potential function

d) $\int_C F \cdot dr$ where $F(x,y) = < y^3 x, 3xy^2 >$ and C is the line segment from (-1,0) to (2,3).

$r(t) = \langle -1, 0 \rangle + t\langle 3,3 \rangle = (-1+3t, 3t)$
$x = -1+3t$, $y = 3t$

$$\int_0^1 (3t)^3(-1+3t)\cdot 3\,dt + 3(-1+3t)(3t)^2 \cdot 3\,dt = \underline{\frac{621}{10}}$$

e) $\int_C y^3 dx + (x^3 + 3xy^2) dy$ where C is the path from (0,0) to (1,1) along the graph of $y = x^3$ and from (1,1) to (0,0) along the graph of $y = x$.



Green:
$$\iint_R 3x^2 + 3y^2 - 3y^2\, dA = 3\int_0^1\int_{x^3}^{x} x^2\, dy\, dx = \underline{1/4}$$

f) $\iint_S \vec{F}\cdot\vec{n}\, dS$ where $F(x,y,z) = < x, y, z >$ and S is $x^2 + y^2 + z^2 = 4$

<u>Gauss</u>
$$\iint \vec{F}\cdot\vec{n}\, dS = \iiint \operatorname{div}(\vec{F})\, dV = \iiint 3\, dV = \underline{3\cdot\frac{4}{3}\pi(2)^3}$$

8. Green's Theorem

a) Use Green's theorem to find $\int_C F \cdot dr$ where $F(x,y) = < y^3, x^3 + 3xy^2 >$ and C is the circle with radius 3, oriented counter-clockwise (You may need the double-angle formula for cos somewhere during your computations)

$$\iint_R 3x^2 + 3y^2 - 3y^2 \, dA = \int_0^{2\pi}\int_0^3 3r^2\cos^2\theta \; r\,dr\,d\theta = \frac{243}{4}\pi$$

b) Evaluate $\iint_R dA$ where R is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ by using a vector field $F(x,y) = < -\frac{y}{2}, \frac{x}{2} >$ and the boundary C of the ellipse R.

$r(t) = \langle 2\cos(t), 3\sin(t) \rangle, \; t \in [0, 2\pi]$

$$\iint \frac{1}{2} - \left(-\frac{1}{2}\right) dA = \int \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle d\vec{r} = \frac{1}{2}\int -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} -3\sin(t)\cdot(-2\sin(t)) + 2\cos(t)\cdot 3\cos(t)\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} 6\, dt = \underline{6\pi}$$

9. Evaluate the following integrals. You can use any theorem that's appropriate:

c) $\int_C 2xyz\,dx + x^2 z\,dy + x^2 y\,dz$ where C is a smooth curve from (0,0,0) to (1,4,3)

Already done above

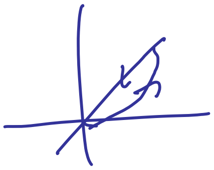
d) $\int_C y\,dx + 2x\,dy$ where C is the boundary of the square with vertices (0,0), (0,2), (2,0), and (2,2)

<u>Green's</u> $\iint_R 2 - 1\, dA = \text{area (square)} = \underline{4}$

e) $\int_C xy^2\,dx + \underline{x^2 y\,dy}$, where C is given by $r(t) = < 4\cos(t), 2\sin(t) >$, t between 0 and 2 Pi.

<u>Green</u> $\iint_R 2xy - 2xy \, dA = 0$ (also: closed curve + conservative vector field

f) $\int_C xy\,dx + x^2 dy$ where C is the boundary of the region between the graphs of $y = x^2$ and $y = x$.



Green $\iint 2x - x \, dA = \int_0^1 \int_{x^2}^{x} x \, dy\,dx = \frac{1}{12}$

10. Prove the following:

a) If $F(x, y, z) = < M(x, y, z), N(x, y, z), P(x, y, z) >$ is any vector field where M, N, P are twice continuously differentiable then $div(curl(F)) = 0$

first work out all these partials

b) A function (not a vector field) $f(x, y, z)$ is called harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$. Show that for any function $f(x, y, z)$ the function $\frac{1}{f(x, y, z)}$ is harmonic.