Panel 1

Last Time

Function $f(x,y)$
- $\int_\gamma f(x,y)\,ds$
- $\int_\gamma f(x,y)\,dx$
- $\int_\gamma f(x,y)\,dy$

Vector field $\vec{F} = \langle M, N\rangle$ – $\int \vec{F}\,d\vec{r} = \int M\,dx + N\,dy$

Fundamental Thm. of Line Integration:

1

Panel 2

Fundamental Theorem for Line Integrals

If $\vec{F}$ is conservative with potential function f, and $\gamma(t)$, $a \le t \le b$, a smooth curve. Then:

$$\int_\gamma \vec{F}\,d\vec{r} = f(\gamma(b)) - f(\gamma(a))$$

Consequences: If $\vec{F}$ is conservative then:

① $\int_{\gamma_1} \vec{F}\,d\vec{r} = \int_{\gamma_2} \vec{F}\,d\vec{r}$ for any two curves with same start/end point

② $\oint_\gamma \vec{F}\,d\vec{r} = 0$ for any closed curve γ.

2

Panel 3

$\vec{F}(x,y) = \langle x^2, y^2 \rangle$, C parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$

$\langle t, 2t^2 \rangle$

$$\int_C \vec{F}\,d\vec{r} = \int_C x^2\,dx + y^2\,dy =$$

(1)
$$= \int_{-1}^{2} t^2 \cdot 1\,dt + (2t^2)^2\, 4t\,dt = \int_{-1}^{2} t^2 + 16t^5\,dt = \frac{1}{3}t^3 + \frac{16}{6}t^6 \Big|_{-1}^{2} =$$

$$= \frac{1}{3}8 + \frac{\cancel{16}\,8}{3}\cdot 64 - \left(-\frac{1}{3}\right) - \frac{16}{6} = \frac{8 + \cancel{16}\,8\,64 + 1 - 8}{3}$$

(2) let $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$ is potential function.

$$\Rightarrow \int_C x^2\,dx + y^2\,dy = \frac{1}{3}(x^3 + y^3)\Big|_{(-1,2)}^{(2,8)} = \frac{1}{3}8 + \frac{1}{3}512 - \left(-\frac{1}{3}\right) - \frac{8}{3} =$$

$$= \frac{8 + 512 + 1 - 8}{3}$$

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Panel 4

$\vec{F} = \langle yz, xz, xy + 2z \rangle$

$C(z)$

$f_x = yz \Rightarrow f = xyz + C(y,z)$

$f_y = xz + C_y(y,z) = xz \Rightarrow C_y(y,z) = 0 \Rightarrow C(y,z) = C(z)$

$f = xyz + C(z)$

$f_z = xy + C'(z) = xy + 2z \Rightarrow C(z) = z^2 + c$

$f(x,y,z) = xyz + z^2$

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Panel 5

Ex: Let $F(x,y) = \langle \frac{y^2}{1+x^2}, 2y\arctan(x) \rangle$ and

$r(t) = \langle t^2, 2t \rangle$, $t \in [0,1]$. Find $\int_\gamma \vec{F}\, d\vec{r}$

① $\int_C F\,dr = \int \frac{y^2}{1+x^2}\, dx + 2y \arctan(x)\, dy = f((1,2)) - f(0,0) = 4\arctan(1)$

too hard $= \int \frac{(2t)^2}{1+t^4} 2t + 2\cdot 2t \arctan(t^2)\, 2\, dt$

What $f_x = \frac{y^2}{1+x^2} \Rightarrow f = y^2 \cdot \arctan(x)$

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Panel 6

Ex: Find $\oint_\gamma \tan(y)\, dx + x\sec^2(y)\, dy = 0$

where $\gamma(t) = \langle \cos(t), \sin(t) \rangle$, $t \in [0, 2\pi]$

because γ is closed and F is conservative.

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Panel 7

Summary of Conservative Vector Field

$\vec{F} = \nabla f$, f is potential function

$\Leftrightarrow \int_C \vec{F}\, d\vec{r}$ is independent of path from A to B

$\Leftrightarrow \oint_C \vec{F}\, d\vec{r} = 0$ for all closed curves C

Thm: If domain is simply connected and all partials are continuous, then

$\text{curl}(\vec{F}) = 0 \Leftrightarrow \vec{F}$ conservative

Note: "Simply connected" means "no holes"

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Panel 8

HW

Find a conservative vector field that has the given potential:

$f(z, y, z) = \sin(x^2 + y^2 + z^2)$

Find $div(\nabla \cdot F)$ and $curl(F) = \nabla \times F$

$F(x, y, z) = < x^2 z, y^2 x, y + 2z >$

Evaluate $\int_C (x - y)dx + xdy$ if C is the graph of $y^2 = x$ from (4,-2) to (4,2)

Find the work done by F(x,y,z) along the curve $< t, t^2, t^3 >$ from (0, 0, 0) to (2, 4, 8), where $F(x, y, z) = < y, z, x >$

Check which of the following vector fields is not conservative.

$F(x, y) = < 3x^2 y + 2, x^3 + 4y^3 >$

$F(x, y) = < e^x, 3 - e^x \sin(y) >$

$F(x, y, z) = < 8xz, 1 - 6yz^2, 4x^2 - 9y^2 z^z >$

Show that the line integrals are independent of the path, and find their value:

$$\int_{(-1,2)}^{(3,11)} (y^2 + 2xy)dx + (x^2 + 2xy)dx$$

$$\int_{(1,0,2)}^{(-2,1,3)} (6xy^3 + 2z^2)dx + (9x^2 y^2)dy + (4xz + 1)dz$$

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Panel 9

Can work out $\int \vec{F}\, d\vec{r}$

- if conservative: shortcut
- line integral

$\oint \vec{F}\, d\vec{r}$

- conservative: $= 0$
- line int
- $\Longrightarrow$ Green's Theorem

"sad history"

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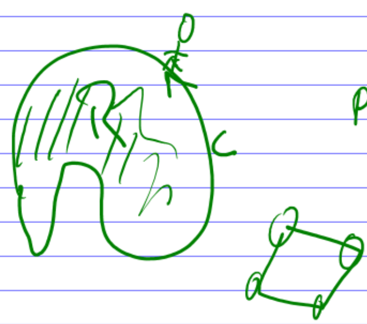
Panel 10

Green's Theorem: R a region in xy-plane with boundary curve C. C is piecewise smooth, non-intersecting, closed, and positively oriented. $\vec{F} = (M, N)$ is a smooth vector field. Then:

$$\oint_C \vec{F}\, d\vec{r} = \oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$



pos. oriented: as you walk along the curve, interior is to the left!

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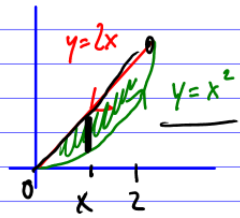
Panel 11

Ex: Evaluate $\oint_C 5xy\,dx + x^3\,dy$, where C is as shown:



Method A: Green's theorem

$$\oint_C 5xy\,dx + x^3\,dy = \iint_R 3x^2 - 5x\,dA =$$

$$= \int \int 3x^2 - 5x\,dy\,dx =$$

$$= \int_0^2 \int_{x^2}^{2x} 3x^2 - 5x\,dy\,dx = -\frac{28}{15}$$

Method B: $\int_0^2 5t\cdot t^2\,dt + (t^2)^3\,2t\,dt = \#$

$(t, 2t)$ $\int_2^0 5t(2t)\,dt + (t)^3\,2\,dt = \#$

$\Rightarrow -\frac{28}{15}$

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Panel 12

Ex: Evaluate $\oint_C 2xy\,dx + (x^2+y^2)\,dy$, C is $4x^2 + 9y^2 = 36$

Think (Greens) $= \iint_R (2x - 2x)\,dA = 0$

(F is conservative)

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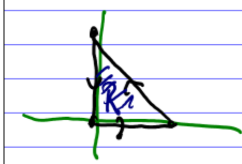
Panel 13

Ex: Find $\oint_{\gamma} \left(x \sin(y^2) - y\right) dx + \left(x^2 y \cos(y^2) + 3x\right)$

where γ is the triangle $(0,0)$, $(1,0)$, $(0,1)$.



by Green: $= \iint_R \cancel{2xy\cos(y^2)} + 3 - \cancel{2yx\cos(y^2)} - 1 \, dA$

$= \iint_R 2 \, dA = 2 \iint_R dA =$

$2 \cdot \text{area}(\text{triangle}) = 2 \cdot \frac{1}{2} = \underline{\underline{1}}$

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Panel 14

Evaluate $\oint_C \left(3y - e^{\sin(x)}\right) dx + \left(7x + \sqrt{y^4+1}\right) dy$

where C is the circle $x^2 + y^2 = 9$

HW

$\iint_{\text{circle}} 7-4 \, dA = 3 \iint_{\text{circle}} dA = 3 \cdot \pi (3)^2 = \underline{27\pi}$

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Panel 15

Theorem: If D is a region enclosed by a curve C

then $\text{area}(D) = \frac{1}{2} \oint_C x\,dy - y\,dx$

$$\oint \overbrace{x}^{N} dy \overbrace{-y}^{M} dx = \iint 1-(-1)\,dA = \iint 2\,dA = 2\,\text{area}(D)$$

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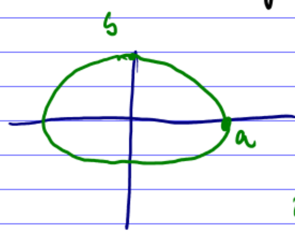
Panel 16

Ex: Find area enclosed by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

ellipse:



$\langle a\cos(t), b\sin(t)\rangle$

$$\text{area} = \frac{1}{2}\oint x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} [a\cos(t)\cdot b\cos(t) - b\sin(t)(-a\sin(t))]\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2(t) + \sin^2(t))\,dt =$$

$$= \frac{1}{2} ab\, 2\pi = \boxed{ab\pi}$$

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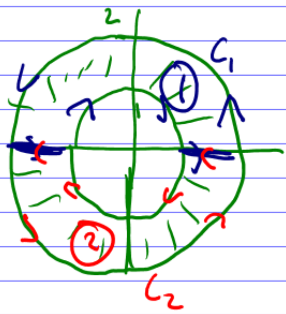
Panel 17

Ex: Evaluate $\oint_C y^2\,dx + 3xy\,dy$ where C is the boundary of the region between $x^2+y^2=1$ and $x^2+y^2=4$.

← upper half plane



$$\oint \overset{M}{y^2}\,dx + \overset{N}{3xy}\,dy =$$

$$\int_{C_1} \_\_ + \cancel{\int_{C_2} \_\_}$$

$$\iint_{1} 3y - 2y\,dA = \iint_{1} y\,dA = \int_0^{\pi}\int_1^2 r\sin(t)\,r\,dr\,dt$$

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Panel 18

15. Evaluate $\int_C 2(x+y)dx + 2(x+y)dy$, C curve from (-2, 2) to (4, 3)

HW

16. Find the work done by the force field $F = \langle 9x^2y^2, 6x^3y - 1 \rangle$ from P(0,0) to Q(5,9)

HW

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Panel 19

18. Evaluate $\oint_C 2xy\,dx + (x+y)\,dy$ where C bounds the region between $y = 0$ and $y = 4 - x^2$.

HW

21. Evaluate $\oint_C x\sin(y^2) - y^2)dx + (x^2\cos(y^2) + 3x)dy$ where C is the boundary of the trapezoid with vertices (0, -2), (1, -1), (1, 1), and (0, 2).

HW

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