Panel 1

Last Time:

Finding potential functions for conservative vector fields $\vec{F} = \langle M, N \rangle$ or $\vec{F} = \langle M, N, P \rangle$

Line Integral of a function with respect to arc length

$$\int_C f(x,y)\,ds = \int f(x(t), y(t)) \sqrt{x'^2 + y'^2}\,dt$$

Line Integral of a function with respect to x or y

$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\, x'(t)\,dt$$

Integral of a vector field $\vec{F}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy =$$

1

Panel 2

Integral Soup

$\int_a^b f(x)\,dx$ "net" area under curve

$\int_C f(x,y)\,ds =$

$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy$

$\int_C f(x,y)\,dx$

$\iiint_Q f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dx\,dy\,dz$

$\int_C f(x,y)\,dy$

$\int_C ds = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$ arc length

$\int_C \vec{F} \cdot d\vec{r}$

$\iint_R ds$ X

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Panel 3

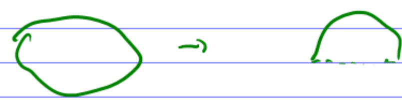
Let $f(x,y) = x^2 - xy + y^2$, $F(x,y) = <2x - y, 2y - x>$, $D = \{(x,y) : x^2 + y^2 \le 1\}$, $C = \{(x,y) : x^2 + y^2 = 1, y \ge 0\}$, $\gamma_1(t) = <t, 0>$, $t \in [-1,1]$, and $\gamma_2(t) = <t, \sin(\pi t)>$, $t \in [-1,1]$.

1. Sketch each object

f is "sheet" of bent paper

F is "river of streaming water"

D is a "filled-in set"  or blob

C boundary of a blob 

γ_1, γ_2 are curve with direction

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Panel 4

Let $f(x,y) = x^2 - xy + y^2$, $F(x,y) = <2x - y, 2y - x>$, $D = \{(x,y) : x^2 + y^2 \le 1\}$, $C = \{(x,y) : x^2 + y^2 = 1, y \ge 0\}$, $\gamma_1(t) = <t, 0>$, $t \in [-1,1]$, and $\gamma_2(t) = <t, \sin(\pi t)>$, $t \in [-1,1]$.

a) $\iint_D f(x,y)dA$ Nope

b) $\cancel{\int_D f(x,y)ds}$

c) $\int_C f(x,y)ds$ ✓ (two answers because of missing orientation

d) $\int_{\gamma_1} f(x,y)dx$ ✓

e) $\int_{\gamma_2} f(x,y)dy$ ✓

f) $\cancel{\int_{\gamma_1} f(x,y)dr}$

g) $\cancel{\int_{\gamma_1} F(x,y)dx}$

h) $\cancel{\int_D F(x,y)dr}$

i) $\int_C F(x,y)dr$ ✓ (two answers)

j) $\int_{\gamma_1} F(x,y)dr$ ✓

k) $\int_{\gamma_2} F(x,y)dr$ ✓

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Panel 5

Let $g(x) = x^2 + e^x$, $f(x,y) = \boxed{x y e^{x^2}}$, $\vec{F}(x,y) = \langle xy, x^2 - y^2 \rangle$

C: $r(t) = \langle \boxed{2t+1}, t^3 - 3 \rangle$, $t \in [0,1]$, $R = [0,1] \times [0,1]$. Setup:

① Area under g from 0 to 1

$$\int_0^1 \underbrace{x^2 + e^x}_{\geq 0} dx$$

② Length of curve C

$$\int_0^1 \sqrt{2^2 + (3t^2)^2}\, dt$$

③ Volume under f over R

$$\int_0^1 \int_0^1 x y e^{x^2}\, dy\, dx$$

④ Surface area of f over R

$\times^2$

⑤ Work through $\vec{F}$ along C

$$\int_C \vec{F}\, d\vec{r} = \int_C \langle xy, x^2 - y^2 \rangle \cdot \langle dx, dy \rangle$$

⑥ Area of "curtain" over C under f

$$\int_C f\, ds = \int_0^1 \underbrace{(2t+1)}_{x} \underbrace{(t^3-3)}_{y} e^{(2t+1)^2} \sqrt{2^2 + (3t^2)^2}\, dt$$

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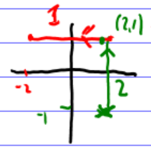
Panel 6

Paths:

Line segment from $(3,1)$ to $(1,1)$:

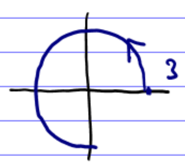
$$\ell(t) = \langle 3, 1 \rangle + t \langle -2, 0 \rangle = \langle 3 - 2t, 1 \rangle$$



$\ell_1(t) = \langle 2, t \rangle$, $t \in [-1, 1]$

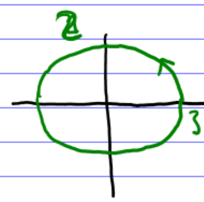
$= \langle 2, -1 \rangle + t \langle 0, 2 \rangle = \langle 2, -1 + 2t \rangle$, $t \in [0, 1]$

$\ell_2(t) = \langle t, 1 \rangle$, $t \in [-2, 2]$



$r(t) = \langle 3\cos(t), 3\sin(t) \rangle$

$t \in [0, \frac{3\pi}{2}]$



$t \in [0, 2\pi]$

$r(t) = \langle 3\cos(t), 2\sin(t) \rangle$

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Panel 7

Fundamental Theorem of Line Integration

Suppose $\vec{F}$ is a conservative vector field. Then

$$\int_C \vec{F}\, d\vec{r} = f(b) - f(a), \quad f \text{ is potential of } \vec{F}$$

Proof

$$\int_C \vec{F}\, d\vec{r} = \int_C \langle M, N\rangle \langle dx, dy\rangle = \int_C M\,dx + N\,dy = \int_C f_x\,dx + f_y\,dy =$$

$$= \int_C \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}\,dt + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\,dt = \int \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\,dt$$

$$= \int_a^b \frac{\partial f}{\partial t}\,dt = f(b) - f(a)$$

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Panel 8

Ex: Find work done by gravitational field

$$\vec{F}(\vec{r}) = -\frac{mMG}{\|r\|^3}\vec{r}$$ moving particle from

$(3,4,12)$ to $(2,2,0)$.

$$\int_C \vec{F}\cdot d\vec{r},$$

Note that curve is not specified. F needs to be conservative so I can do the problem

$$\vec{F} = -\frac{mMG}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$$ has potential function:

$$f(x) = mMG\,(x^2+y^2+z^2)^{-1/2}$$ is potential because $\nabla f = \vec{F}$

$$\Rightarrow \int_C \vec{F}\,d\vec{r} = mMG(x^2+y^2+z^2)^{-1/2}\Big|_{(3,4,12)}^{(2,2,0)} = mMG\left[\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{169}}\right]$$

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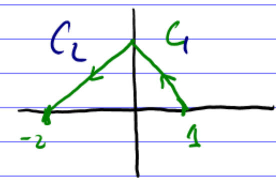
Panel 9

Evaluate $\int_C \overset{M}{y^2}\,dx - \overset{N}{2x}\,dy$.



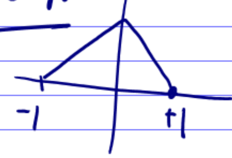
$\int_{C_1} y^2\,dx - 2x\,dy + \int_{C_2} y^2\,dx - 2x\,dy$ C_1: $l(t) = \langle t, 1-t\rangle$, $t \in [0,1]$

Alternatively, try Fund Thm. C_2: $l(t) = \ldots$

$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ not, can not use Fund. Thm.

Panel 10

$\int_C \overset{M}{2x+y^2}\,dx + \overset{N}{2xy}\,dy$ $F = \langle 2x+y^2, 2xy\rangle$



$\frac{\partial N}{\partial x} = 2y = \frac{\partial M}{\partial y}$ ✓ Is conservative, so need potential.

$f_x = 2x+y^2 \Rightarrow f = x^2+xy^2+C(y)$

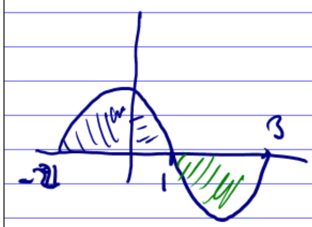
$f_y = 2xy + C'(y) = 2xy \Rightarrow C'(y) = 0, \quad C = c$

$f(x,y) = x^2+xy^2+c$

$\int F\,dr = \int 2x+y^2\,dx + 2xy\,dy = \left(x^2+xy^2\right)\Big|_{(1,0)}^{(-1,0)} = 0 - 1 = \underline{\underline{-1}}$

Panel 11

Hmm. The integral was negative. Why?



$$\int_{-2}^{0} f(x)\,dx = \text{positive}$$

$$\int_{-2}^{1} f(x)\,dx = \text{positive}$$

$$\int_{-2}^{3} f(x)\,dx = \text{zero}$$

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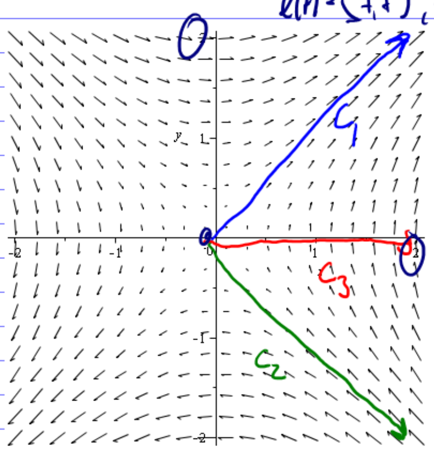
Panel 12

Ex: Are the following integrals positive or negative?

$r(t) = \langle t, t \rangle,\ t \in [0,2]$



$$\int_{C_1} \vec{F}\,d\vec{r} = \int_C y\,dx + x\,dy$$

positive $= \int_0^2 t\,dt + t\,dt = \int_0^2 2t\,dt = 4$

$$\int_{C_2} \vec{F}\,d\vec{r} = \int_0^2 -t\,dt + t\,(-1)\,dt = -4$$

$r_2(t) = \langle t, -t \rangle,\ t \in [0,2]$

$$\int_{C_3} \vec{F}\,d\vec{r} = \int_0^2 0\cdot dt + t\,0\,dt = 0$$

$r_3(t) = \langle t, 0 \rangle,\ t \in [0,2]$

$\vec{F} = \langle y, x \rangle$

$$\int_C \vec{F}\,d\vec{r} = \int_C y\,dx + x\,dy$$

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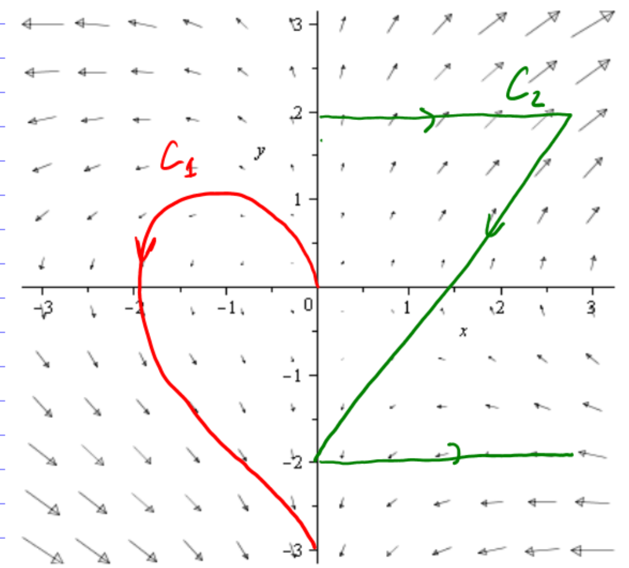
Panel 13

Ex: Are the following integrals positive or negative?



$\int_{C_1} \vec{F}\, d\vec{r}$ POS

pos or neg

$\int_{C_2} \vec{F}\, d\vec{r}$ zero?

pos or neg

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Panel 14

Which of the following vector fields look conservative?

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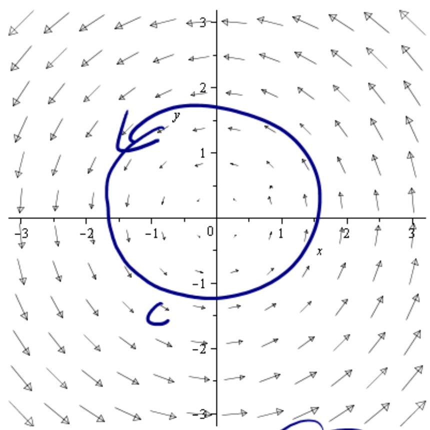
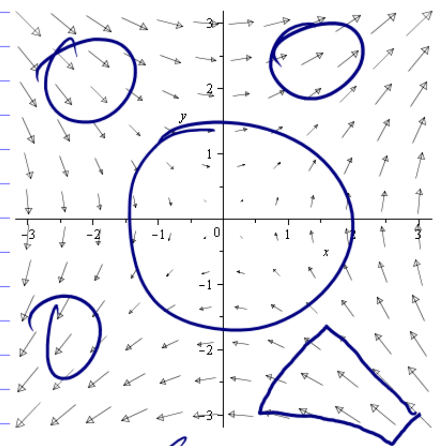
Panel 15

Which of the following vector fields looks conservative?





$(-y, x)$

$\oint_C \vec{F}\, d\vec{r} > 0$,

$\vec{F}$ not conservative

(y, x)

$\vec{F}$ could be conservative

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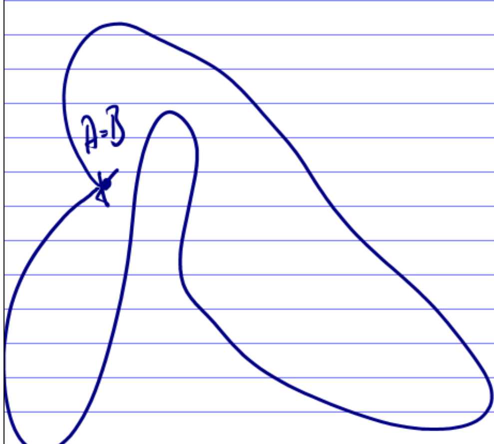
Panel 16

Corollary 2: If $\vec{F}$ is conservative and C a closed curve then $\int_C \vec{F}\, d\vec{r} = 0$ $= f(B) - f(A) = 0$

Note: if C is closed we write $\oint_C \vec{F}\, d\vec{r}$



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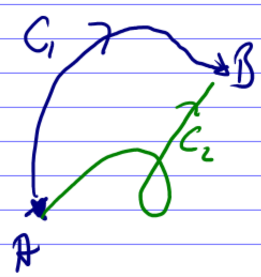
Panel 17

Thm: If $\vec{F} = \nabla f$, i.e. $\vec{F}$ is conservative and C is a smooth curve from A to B then

$$\int_C \vec{F}\, d\vec{r} = f(B) - f(A)$$

Corollary:



$$\int_{C_1} \vec{F}\, d\vec{r} = \int_{C_2} \vec{F}\, d\vec{r}$$

If $\vec{F}$ is conservative, then work does not depend on the curve from A to B

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Panel 18

Corollary 2: If $\vec{F}$ is conservative and C a closed curve then

$$\int_C \vec{F}\, d\vec{r} = 0$$

Note: if C is closed we write $\oint_C \vec{F}\, d\vec{r}$

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Panel 19

Name: ____________

Quiz #8

1. Find potential for $F(x,y) = \langle y\cos(x) + 2xy,\ \sin(x) + x^2 \rangle$

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Panel 20

2. Let $f(x,y) = x^2 y$ and $\vec{r}(t) = \langle 2t, 3t-1 \rangle$, $t \in [0,1]$

a) $\int_C f(x,y)\,ds$

b) $\int_C f(x,y)\,dx$

c) $\int_C f(x,y)\,dy$

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