Panel 1

Last Time:

$\vec{F} = \langle M, N \rangle$ conservative $\Rightarrow$ $N_x = M_y$

$\vec{F} = \langle M, N, P \rangle$ conservative $\Rightarrow$ $\text{curl}(\vec{F}) = 0$

How to find potential functions

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt$$

C a curve $(x(t), y(t))$, $t \in [a,b]$

"geometric meaning"

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Panel 2

(1) If $f: \mathbb{R}^3 \to \mathbb{R}$ is a function and $F: \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, which expression is meaningful:

curl (f) X

grad (f) ✓

div (F) ✓

curl (grad (f)) ✓

grad (F) X

grad (div (F)) ✓

div (grad (f)) ✓

grad (div (f)) X

curl (curl (F)) ✓

div (div (F)) X

(grad (f)) × (div (F)) X

grad (div (curl (grad (f))))

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Panel 3

Find potential for $F(x,y) = \langle y\cos(x) + 2xy, \underline{\sin(x) + x^2} \rangle$

① $N_x = \cos(x) + 2x \overset{?}{=} M_y = \cos(x) + 2x$

② $f_x = y\cos(x) + 2xy \xRightarrow{\text{int. w.r. to } x} f = y\sin(x) + x^2 y + C(y)$

$\partial/\partial y$

$f_y = \sin(x) + x^2 + C'(y) = \sin(x) + x^2 \Rightarrow C'(y) = 0$

$C = \text{const.}$

$f(x,y) = y\sin(x) + x^2 y + C$

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Panel 4

Let $F(x,y,z) = \langle y+z, \underline{x+z}, y\underline{+x} \rangle$. Find potential:

$f_x = y+z \Rightarrow f = xy + xz + C(y,z)$

∂_y

$f_y = \cancel{x} + C_y = \cancel{x} + z \Rightarrow C_y = z \Rightarrow C = yz + D(z)$

$f = xy + xz + yz + D(z)$

∂_z

$f_z = x + y + D'(z) \overset{!}{=} y + x \Rightarrow D' = 0 \Rightarrow D = d$

$f(x,y,z) = xy + xz + yz + d$

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Panel 5

Occasionally it is possible to guess a potential function:

Gravity field: $\vec{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$

Potential function: $f(x,y,z) = -(x^2+y^2+z^2)^{-1/2}$

Works because it does!

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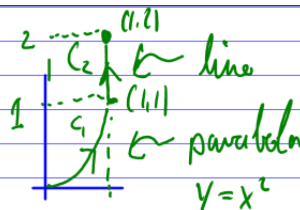
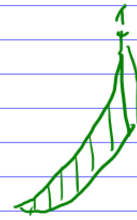
Panel 6

Ex: Evaluate $\int_C 2x\, ds$ where C:



$$\int_C 2x\, ds = \int_{C_1} 2x\, ds + \int_{C_2} 2x\, ds$$



C_1: $r(t) = \langle \overset{=x}{t}, \overset{=y}{t^2} \rangle, t \in [0,1]$:

$$\int_{C_1} 2x\, ds = \int_0^1 2t \cdot \sqrt{1+4t^2}\, dt \quad \checkmark$$

C_2: $r(t) = (1,1) + t(0,1) = \langle \overset{x}{1}, \overset{y}{1+t} \rangle \quad , t \in [0,1]$

$$\int_{C_2} 2x\, ds = \int_0^1 2\sqrt{1}\, dt = \underline{\underline{2}}$$

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Panel 7

Question: What is $\int_C f(x,y)\,ds$ if

a) C goes along x-axis from a to b

b) C goes along y-axis from c to d

a) $r(t) = \langle a + t(b-a), 0 \rangle$, $t \in [0,1]$

$$\int_a^b f(x(t), y(t)) \sqrt{[x'(t)]^2}\, dt = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

b) $$\int_c^d f(x(t), y(t))\, y'(t)\, dt$$

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Panel 8

We also define two variations of line integrals:

C is a curve given by $r(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$

old

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt$$

new

$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))(x')\, dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))(y')\, dt$$

$$\int f(x,y)\,dx$$

$$\int_a^b f(x,y)\,dy$$

intended notation

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Panel 9

Ex: Find $\int_C xy^2\,dx$ and $\int_C xy^2\,dy$ where C is parabola from $(0,0)$ to $(2,4)$ — either one would work

$C: r(t) = \langle t, t^2 \rangle,\ t \in [0,2]$ $\quad \left(r(t) = \langle t, 2t^2 \rangle,\ t \in [0,1] \right)$

$$\int_C xy^2\,dx = \int_0^2 t(t^2)^2 \frac{\partial x}{\partial t}\,dt = \int_0^2 t^5\,dt = \frac{1}{6}2^6$$

$$\int_C xy^2\,dy = \int_0^2 t(t^2)^2 \frac{\partial y}{\partial t}\,dt = \int_0^2 t^5 \cdot 2t\,dt = 2\int_0^2 t^6\,dt = \frac{2}{7}2^7$$

(much easier than $\int \_\, ds$)

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Panel 10

Def: C a curve $r(t) = \langle x(t), y(t) \rangle,\ t \in [a,b]$

$$\Rightarrow \int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x')^2 + (y')^2}\,dt$$

(line integral of f along curve C)

$$\Rightarrow \int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'\,dt$$

(line integral of f along curve C with respect to x)

$$\Rightarrow \int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'\,dt$$

(line integral of f along curve C with respect to y)

$$\Rightarrow \int_C f(x,y)\,dx + g(x,y)\,dy = \int_C f\,dx + \int_C g\,dy$$

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Panel 11

<u>Ex:</u> Evaluate $\int_C xy\,dx + x^2\,dy$ if starts at. $(2,1)$ ends at. $(4,5)$

$C_1: \langle 3t-1,\ 3t^2-2t\rangle,\ 1 \le t \le \boxed{5/3}$

C_2: line segment from $(2,1)$ to $(4,5)$

$$\int_{C_1} xy\,dx + x^2\,dy = \int_{C_1} xy\,dx + \int_{C_1} x^2\,dy =$$

$$= \int_1^{5/3} (3t-1)(3t^2-2t)\,3\,dt + \int_1^{5/3} (3t-1)^2 \underbrace{(6t-2)}_{y'}\,dt =$$

$$\frac{75}{4} + \frac{85}{2}$$

$$\int_{C_2} (2+2t)(1+4t)\,2\,dt + \int_{C_2} (2+2t)^2\,4\,dt = \text{HW}$$

$C_2: r(t) = \langle 2,1\rangle + t\langle 2,4\rangle = \langle 2+2t,\ 1+4t\rangle$

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Panel 12

<u>Ex:</u> Find $\int_C y^2\,dx + x\,dy$ where (a) C_1 is line from $(-5,-3)$ to $(0,2)$ and (b) C_2: $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$

I return!

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Panel 13

Line Integrals of Vector Fields:

Suppose F is a vector field on a smooth curve C, defined via $\vec{r}(t)$, $a \le t \le b$. Then the line integral of F along C is:

$$\int_C \vec{F} \overset{\text{dot}}{\cdot} d\vec{r} = \int_C \vec{F}(x(t), y(t)) \cdot \underline{r'(t)\, dt}$$

$$\int \vec{F} \cdot d\vec{r} = \int_C \langle M, N \rangle \cdot \langle dx, dy \rangle =$$

$$= \int_C M\,dx + N\,dy = \underline{\text{know}}$$

$$= \int_C M(x(t), y(t))\, \underline{x'\,dt} + \int N(x(t), y(t))\, \underline{y'}\,dt$$

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Panel 14

Ex: Let $\vec{F}(x,y) = \langle \underline{x^2}, \underline{-xy} \rangle$ C quarter circle radius 1.

Find $\int_C \vec{F}\, d\vec{r}$

C: $r(t) = \langle \cos(t), \sin(t) \rangle$ $t \in [0, \pi/2]$

$$\int_C \vec{F}\, d\vec{r} = \int_C x^2 \underline{dx} - xy\,dy =$$

$$-\int_0^{\pi/2} \cos^2(t)\sin(t)\,dt - \int_0^{\pi/2} \cos(t)^2 \sin(t)\,dt =$$

$$= -2\int_0^{\pi/2} \cos^2(t)\sin(t)\,dt = \underline{\underline{-2/3}}$$

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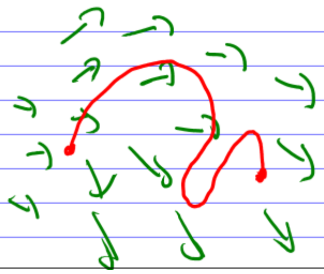
Panel 15

Physics Interpretation of Line Integral

$$\int_C \vec{F}\, d\vec{r} = \int_a^b M dx + N dy \, (+ P dz)$$

is the work done by a vector field $\vec{F}$ to move a particle from start to end along curve C.



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Panel 16

Ex. Find work required to move particle from $(2,0,0)$ to $(3,4,5)$ through force field $\vec{F} = \langle \overset{M}{y}, \overset{N}{z}, \overset{P}{x} \rangle$ along straight line

$$r(t) = \langle 2,0,0\rangle + t\langle 1,4,5\rangle = \langle \overset{x}{2+t}, \overset{y}{4t}, \overset{z}{5t}\rangle \quad , t \in [0,1]$$

$dx = 1dt$

$$\int_C \vec{F}\, d\vec{r} = \int_C M dx + N dy + P dz = \int_C y\, dx + z\, dy + x\, dz$$

$$= \int_0^1 4t \cdot dt + 5t \cdot 4 dt + (2+t) 5 dt =$$

$$= \int_0^1 29t + 10 dt = \underline{\qquad}$$

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Panel 17

Fundamental Theorem of Line Integrals:

Suppose $\vec{F}$ is a conservative vector field. Then

$$\int_C \vec{F}\, d\vec{r} = \text{Mystery} \ldots$$

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