Panel 1

Birds - Eye View so far

$f: \mathbb{R} \to \mathbb{R}$ Calc 1 + 2

$f: \mathbb{R} \to \mathbb{R}^2$ space curves: $r(t) = \langle f(t), g(t) \rangle$

$\left.\begin{array}{l} f: \mathbb{R}^2 \to \mathbb{R} \\ \\ f: \mathbb{R}^3 \to \mathbb{R} \end{array}\right\}$ functions $z = f(x,y) = x^2 + y^2$ limits, cont., deriv, integration

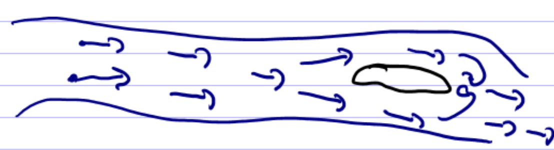
$\left.\begin{array}{l} f: \mathbb{R}^2 \to \mathbb{R}^2 \\ f: \mathbb{R}^3 \to \mathbb{R}^2 \\ f: \mathbb{R}^2 \to \mathbb{R}^3 \\ f: \mathbb{R}^3 \to \mathbb{R}^3 \end{array}\right\}$ <u>Vector Fields</u>

1

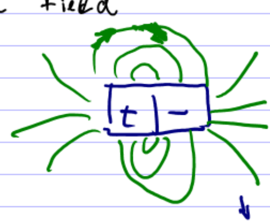
Panel 2

Vector Fields: If for each point P in a region R there is a unique vector having initial point P, then the totality of such vectors is called a vector field

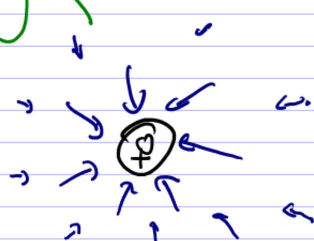
Ex: Flow of water



Ex: Magnetic Field



Ex: Gravity



2

Panel 3

f, g
$\vec r, \vec s$

Mathematically, a vector field is given as:

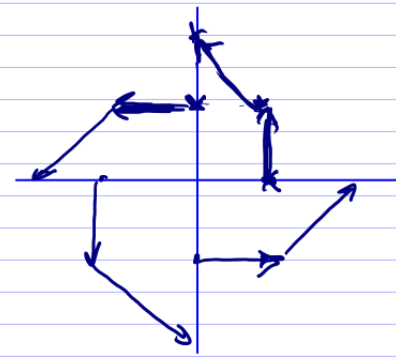
$$\vec F(x,y) = \langle M(x,y), N(x,y) \rangle = M\vec i + N\vec j$$

$$\vec F(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$$

Ex: Describe $F(x,y) = \langle -y, x \rangle = -y\vec i + x\vec j$

points →

(x,y)	$F(x,y)$
$(0,1)$	$\langle -1,0 \rangle$
$(1,0)$	$\langle 0,1 \rangle$
$(-1,0)$	$\langle 0,-1 \rangle$
$(1,1)$	$\langle -1,1 \rangle$
$(1,2)$	
$(0,2)$	



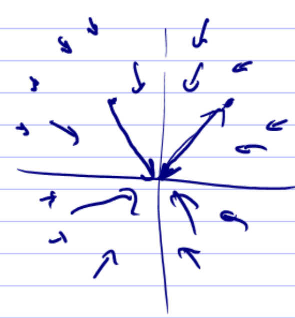
3

Panel 4

Def: If $r(x,y,z) = \langle x,y,z \rangle$ then $F(x,y,z) = \frac{c}{\|r\|^2}\vec u = \frac{c}{\|r\|^3}\vec r$ where $u = r/\|r\|$ is called inverse square field.

Ex: Describe inverse square field for $c = -1$.

$$\vec F(x,y,z) = \frac{-1}{(x^2+y^2+z^2)^{3/2}} \langle x,y,z \rangle$$



a) At each point $\langle x,y,z \rangle$, there is a vector pointing back to the origin

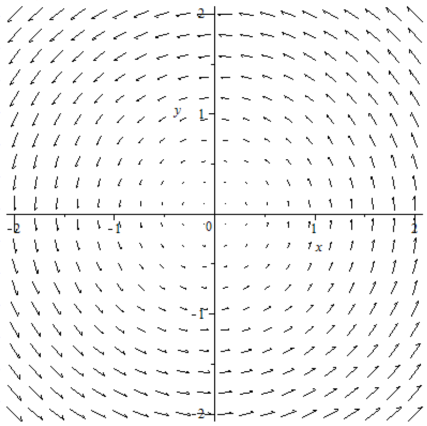
b) length is shorter if you are far away, and big if you are close to $\langle 0,0,0 \rangle$

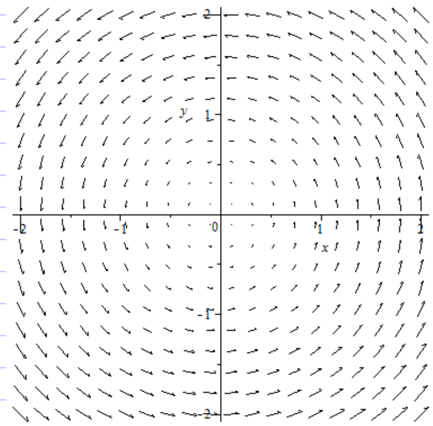
4

Panel 5

Maple offers "fieldplot" and "fieldplot3d"

```
> with(plots);
> fieldplot([-y, x], x = -2..2, y = -2..2);
> fieldplot3d([-x/(x^2+y^2+z^2)^(3/2), -y/(x^2+y^2+z^2)^(3/2), -z/(x^2+y^2+z^2)^(3/2)], x = -2..2, y = -2..2, z = -2..2);
>
```



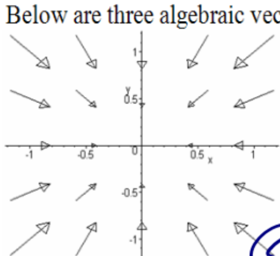


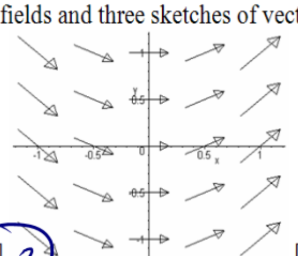
5

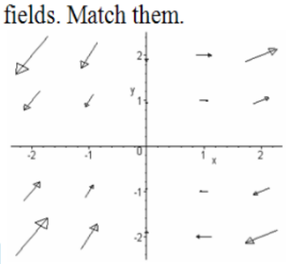
Panel 6

Quiz 8 - Part 1

1. Below are three algebraic vector fields and three sketches of vector fields. Match them.

[A] 

[B] = 2 

[C] 

(1) $F(x,y) = < xy, y(x-1) >$ C

(2) $F(x,y) = < 1, x >$ = B

(3) $F(x,y) = < -x, -y >$ = A

6

Panel 7

Def: Suppose $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$

Then
$$\text{curl}(\vec{F}) = \langle P_y - N_z, -(P_x - M_z), N_x - M_y \rangle$$
$$\text{div}(\vec{F}) = M_x + N_y + P_z$$

(limits, cont., derivatives are applicable to component functions)

To remember, let $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$\Rightarrow \text{div}(\vec{F}) = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle M, N, P \rangle = M_x + N_y + P_z$

$\text{curl}(\vec{F}) = \underline{\nabla \times \vec{F}}$

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Panel 8

Ex: Let $F(x,y,z) = \langle xy, yz, xz \rangle$. Then

curl (F):
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{vmatrix} = \langle 0 - y, -(z - 0), 0 - x \rangle = \underline{\underline{\langle -y, -z, -x \rangle}}$$

(vector field)

$\text{div}(F) = \nabla \cdot \vec{F} = y + z + x$ (function)

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Panel 9

Ex: $F(x,y,z) = \langle xy^2z^4,\ 2x^2y+z,\ y^3z^2 \rangle$

Find curl (F) and div (F)

div $\nabla \cdot \vec{F} = y^2z^4 + 2x^2 + 2y^3z$

$$\text{curl}(F) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy^2z^4 & 2x^2y+z & y^3z^2 \end{vmatrix}$$

$$= \langle 3y^2z^2 - 1,\ -(0 - 4xy^2z^3),\ 4xy - 2xyz^4 \rangle$$

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Panel 10

2. Suppose that $F(x,y,z) = \langle x^3z, x^2z, xy \rangle$ is some vector field.

   a) Find $div(F)$

   HW

   b) Find $curl(F)$

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Panel 11

Show that $\nabla \cdot (f\vec{F}) = f(\nabla \cdot \vec{F}) + (\nabla f) \cdot \vec{F}$, $f: \mathbb{R}^3 \to \mathbb{R}$
$F = \langle M, N, P \rangle$

Proof: $\langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle f(x,y,z) M(x,y,z) , f(x,y,z) \cdot N(x,y,z), f \cdot P \rangle$

$\partial_x(fM) + \partial_y(f \cdot N) + \partial_z(fP) =$

$f_x M + f M_x + f_y N + f N_y + f_z P + f \cdot P_z =$

$f(M_x + N_y + P_z) + (f_x M + f_y N + f_z P) =$

$= f(\nabla \cdot \vec{F}) + \langle f_x, f_y, f_z \rangle \cdot \langle M, N, P \rangle$

$= f(\nabla \cdot \vec{F}) + (\nabla f) \cdot \vec{F}$

qed

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Panel 12

Def: A vector field $\vec{F}$ is conservative if there is a function $f: \mathbb{R}^3 \to \mathbb{R}$ s.t. $\nabla f = \vec{F}$
(f is similar to antiderivative). f is potential function of $\vec{F}$

Ex: Find vector field with potential

$f(x,y,z) = x^2 - 3y^2 + 4z^2$

$F = \nabla f = \langle 2x, -6y, 8z \rangle$

is a conservative vector field with potential function $f = x^2 - 3y^2 + 4z^2$

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Panel 13

Which of the following vector field(s) has as <u>potential function</u> $f(x,y,z) = x^2y^2z^2 + xy + zy$

(a) $\vec{F} = \langle 2x, y, z \rangle$

(b) $\vec{F} = \langle 2xy^2z^2 + x + y \rangle$

(c) $\vec{F} = \langle 2xy^2z^2, y, z \rangle$

(d) $\vec{F} = \langle 2xy^2z^2, x, y \rangle$

(e) $\vec{F} = \langle 2xy^2z^2 + y, 2x^2z^2y + x + z, 2x^2y^2z + y \rangle$

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Panel 14

Suppose $\vec{F}$ is conservative, i.e. $\nabla f = \langle f_x, f_y, f_z \rangle = \vec{F} = (M, N, P)$

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle$$

$= \langle 0, 0, 0 \rangle$ if f has 2nd order partials that are continuous

<u>Theorem:</u> If $\vec{F}$ is conservative in $\mathbb{R}^3$ then <u>$\operatorname{curl}(\vec{F}) = 0$</u>

If $\vec{F}$ is conservative in $\mathbb{R}^2$ then <u>$N_x = M_y$</u>

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Panel 15

Which of the following vector fields is not conservative

(a) $F(x,y) = \langle x, y \rangle$ ✓

(b) $F(x,y) = \langle x^2+y^2, 2xy \rangle$ ✓

(c) $F(x,y) = \langle e^x \cos(y), -e^x \sin(y) \rangle$ ✓

(d) $F(x,y) = \langle x^2 \cos(y), -y^2 \sin(x) \rangle$ ✗

is $N_x = M_y$

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Panel 16

Find potential function for $\vec{F} = (3+2xy, \underline{x^2-3y^2})$ if exists

(1) Check $N_x = M_y$: $2x = 2x$ ✓

(2) Want $f(x,y)$ s.t. $f_x = \underline{3+2xy}$

$\Rightarrow f(x,y) = \int 3+2xy \, dx = \underline{3x + x^2y + C(y)}$

$\Rightarrow f_y(x,y) = \cancel{x^2} + C'(y) = \cancel{x^2} - 3y^2$

$C'(y) = -3y^2 \Rightarrow \underline{C(y) = -y^3 + C}$

$\Rightarrow f(x,y) = \underline{3x + x^2y - y^3 + C}$

$f \xrightarrow{\text{diff } x} M$

check: $f_x = 3+2xy = M$

$f_y = x^2 - 3y^2 = N$

$f \xrightarrow{\text{diff}} f_y = N$, int, answer int.

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