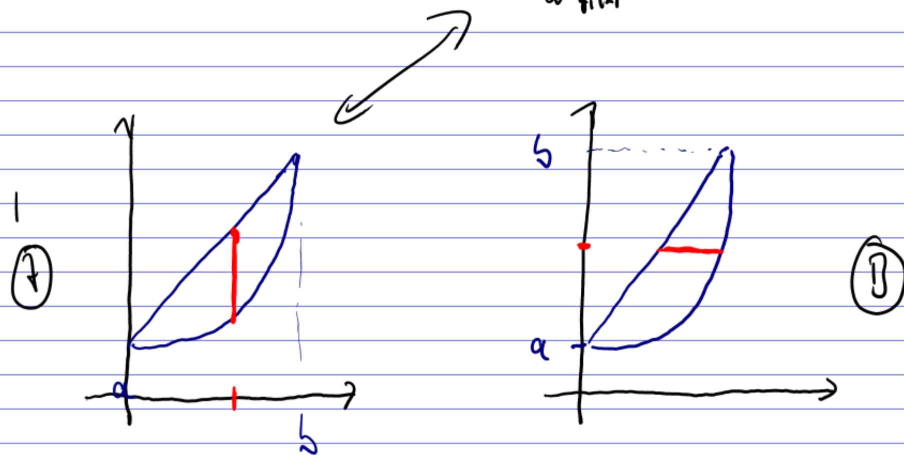
Panel 1

Which picture represents $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$



1

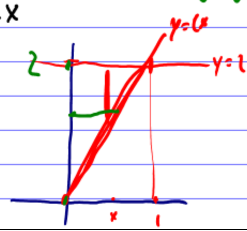
Panel 2

$$\int_0^2\int_0^1 x^2 y\,dx\,dy = \int_0^1\int_0^2 x^2 y\,dy\,dx = \int_0^1 \frac{1}{2}x^2y^2\Big|_{y=0}^{y=2} dx = \int_0^1 2x^2\,dx = \frac{2}{3}$$

$$\int_0^1\int_0^y \sqrt{1-y^2}\,dx\,dy = \int_0^1 x\sqrt{1-y^2}\Big|_{x=0}^{x=y} dy = \int_0^1 y\sqrt{1-y^2}\,dy =$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\left(1-y^2\right)^{3/2}\Big|_0^1 = -\frac{1}{3}(0) - \left(-\frac{1}{3}1\right) = \frac{1}{3}$$

$$\int_0^1 \int_{y=2x}^{y=2} e^{y^2}\,dy\,dx = \int_0^2\int_0^{y/2} e^{y^2}\,dx\,dy = \int_0^2 x e^{y^2}\Big|_{x=0}^{x=y/2} dy = \int_0^2 \frac{1}{2} y e^{y^2}\,dy =$$



$$= \frac{1}{4}\; e^{y^2}\Big|_0^1 = \frac{1}{4}e - \frac{1}{4} = \frac{1}{4}(e-1) > 0$$

2

Panel 3

Tripple Integration

Just like partial derivatives, integration can be extended to higher dimension without problems.

Ex1 $\iiint_R xyz \, dV$, R a cube in 3D, side length 2 (of side length 2)

$$\int_0^2\int_0^2\int_0^2 xyz \, dx\,dy\,dz = \int_0^2\int_0^2 \left[\frac{1}{2}x^2yz \Big|_{x=0}^{2} \right] dy\,dz = 2\int_0^2\int_0^2 yz \, dy\,dz =$$

$$= 2\int_0^2 \frac{1}{2}y^2 z \Big|_{y=0}^{2} dz = 2\cdot 2\int_0^2 z\,dz = 2\cdot 2\cdot 2 = \underline{8}$$

3

Panel 4

Properties of Double Integrals

(1) $\iint_D f(x,y) + g(x,y) \, dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

(2) If $f(x,y) \geq g(x,y)$ then $\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$

(3) $\iint_D 1 \, dA = \text{area}(D)$

(4) If $m \leq f(x,y) \leq M$ then

$$m\,\text{area}(D) = \iint_D m\,dA \leq \iint_D f(x,y)\,dA \leq \iint_D M\,dA = M\,\text{area}(D)$$

4

Panel 5

Ex: Estimate $\iint_D e^{\sin(x)\cos(y)}\, dA$ where D is disk, radius 2

Know $-1 \le \sin(x) \cdot \cos(x) \le 1$

$\Rightarrow \quad e^{-1} \le e^{\sin(x)\cos(y)} \le e^{1}$

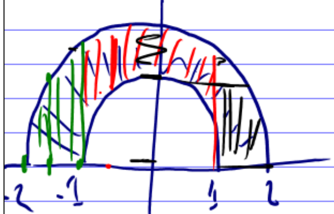
$$\frac{1}{e} \cdot \pi 4 \le \iint_D e^{\sin(x)\cos(y)}\, dA \le e\,\pi 4$$

5

Panel 6

But there are some integral where all tricks (so far) don't work:

$\iint_D (3x + 4y^2)\, dA$ where D is region in upper half plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$



$\iint 3x+4y^2\, dy\, dx = 3$ integrals

$\iint 3x+4y^2\, dx\, dy = 2$ integrals

) refuse!

6

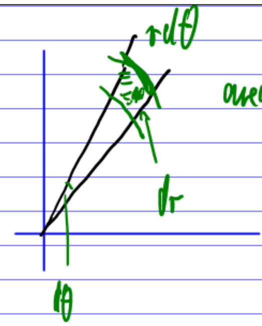
Panel 7

Solution. Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

area $dx\,dy$



area is $\boxed{r\,d\theta\,dr}$

Thm:

$$\iint_D f(x,y)\,dA = \iint_D f(r,\theta)\,r\,dr\,d\theta$$

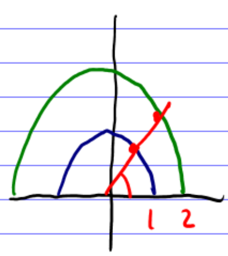
$x = r\cos\theta$

$y = r\sin\theta$

7

Panel 8

$\iint_D 3x^2+3y^2\,dA$, where D is the region in the upper half plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$



$$\iint_D 3x^2+3y^2\,dA = \int_0^{\pi}\int_1^2 3r^2\cos^2\theta+3r^2\sin^2\theta\;r\,dr\,d\theta =$$

$$= \int_0^{\pi}\int_1^2 3r^3\,dr\,d\theta = \int_0^{\pi} \frac{3}{4}r^4\Big|_{r=1}^{2}\,d\theta$$

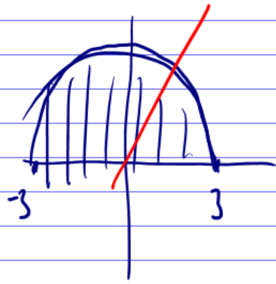
$$= \frac{3}{4}(16-1)\cdot\pi = \frac{45}{4}\pi$$

8

Panel 9

Ex.: $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}=y} \sqrt{x^2+y^2}\,dy\,dx =$

$$\int_0^{\pi}\int_0^{3} \sqrt{r^2}\cdot r\,dr\,d\theta$$



$y=\sqrt{9-x^2}$

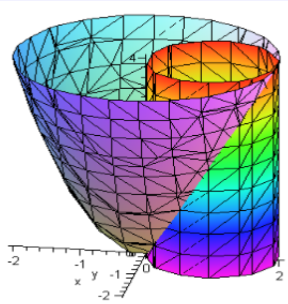
$y^2=9-x^2$

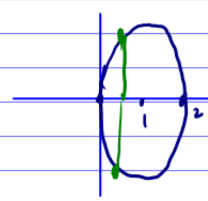
$x^2+y^2=9$

9

Panel 10

Ex: Volume under $z=x^2+y^2$, inside $x^2+y^2=2x$, above xy-plane





$x^2+y^2=2x$

$x^2-2x+1+y^2=0+1$

$(x-1)^2+y^2=1 \Rightarrow y=\sqrt{1-(x-1)^2}$

with(plots):

implicitplot3d({$z=x^2+y^2$, $x^2+y^2=2x$}, $x=-2..2$, $y=-2..2$, $z=0..4$);

$$\int_0^2\int_{-\sqrt{1-(x-1)^2}}^{+\sqrt{1-(x-1)^2}} x^2+y^2\,dy\,dx$$

$x=r\cos(\theta)$

$y=r\sin(\theta)$

$r=2\cos\theta$

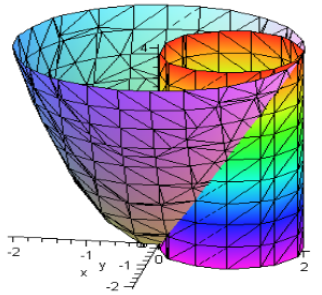
$r^2\cos^2\theta-2r\cos\theta+r^2\sin^2\theta=0$

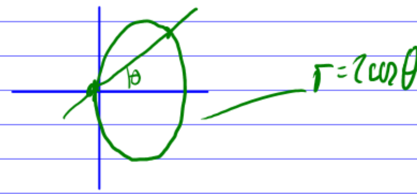
$r^2-2r\cos\theta=0 \Rightarrow r(r-2\cos\theta)=0$

10

Panel 11

Ex: Volume under $z = x^2+y^2$, inside $x^2+y^2 = 2x$, above xy-plane





$$\iint_D x^2+y^2 \, dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos(\theta)} r^2 \, r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{4} r^4 \Big|_0^{2\cos\theta} d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} 4\cos^4(\theta) \, d\theta = \text{look it up.}$$

11