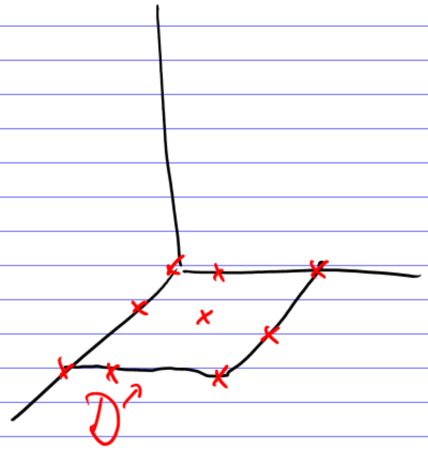
Panel 1

Abs Extrema



① Critical points inside D

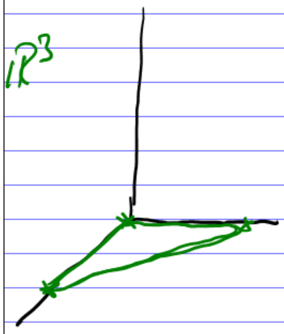
② Critical points on bdry of D

③ Take endpoints

⇒ Pick smallest / largest

Panel 2

Ex: Let $f(x,y) = 3xy - 6x - 3y + 7$. Find abs. extrema over triangle with corners $(0,0)$, $(3,0)$, and $(0,3)$



$f_x = 3y - 6 = 0 \Rightarrow \boxed{y = 2}$

$f_y = 3x - 3 = 0 \Rightarrow \boxed{x = 1}$

Boundary ① $y = 0$, $x \in [0,3]$: $f(x,0) = 6x + 7$

② $x = 0$, $y \in [0,3]$: $f(0,y) = -3y + 7$

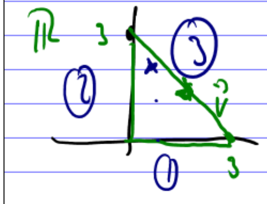
③ $y = -x + 3$, $x \in [0,3]$

$f(x,y) = 3x(-x+3) - 6x - 3(-x+3) + 7$

$= -3x^2 + 9x - 6x + 3x - 9 + 7 = -3x^2 + 6x - 2$

$f'(x) = -6x + 6 \Rightarrow \boxed{x = 1}$ $\boxed{y = 2}$

(x,y)	$f(x,y)$
$(1,2)$	—
$(0,0)$	7
$(0,3)$	—
$(3,0)$	—

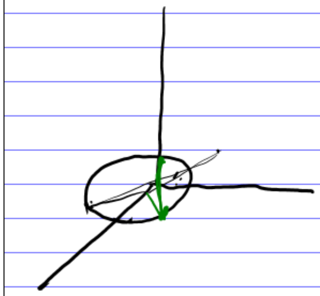


$\vec{v} = \langle 3,-3 \rangle$ or $\langle 1,-1 \rangle$: $r(t) = t\langle 1,-1 \rangle + (0,3)$, $t \in [0,3]$

Panel 3

Find max(abs) of $f(x,y) = x+y$ over unit disk

$$\nabla f = \langle 1,1 \rangle$$



Boundary: $r(t) = \langle \cos(t), \sin(t) \rangle$, $t \in [0, 2\pi]$

$\Rightarrow f(x,y) = f(t) = \cos(t) + \sin(t)$

$f'(t) = -\sin(t) + \cos(t) = 0$, $t = \pi/4$

$\Rightarrow$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Panel 4

Other Method: Lagrange Multipliers

Maximize $f(x,y,z)$ subject to $g(x,y,z) = k$

(Recall: Max. $V = xyz$, given that $2xz + 2yz + xy = 12$)

↙ lambda

① Solve $\nabla f = \lambda \nabla g$ - 3 equations, 4 unknowns

$g(x,y,z) = k$

② Subst. answers into $f(x,y,z)$ and pick largest or smallest

Panel 5

Max: $V = xyz$ subject to $2xz + 2yz + xy = 12$

$\nabla V = \lambda \nabla g$: $yz = \lambda(2z+y)$ $|\cdot x$ $xyz = \lambda x(2z+y)$

$xz = \lambda(2z+x)$ $|\cdot y$ $xyz = \lambda y(2z+x)$

$xy = \lambda(2x+2y)$ $|\cdot z$ $xyz = \lambda z(2x+2y)$

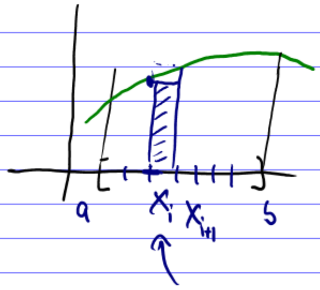
$2xz+2yz+xy=12$

... then some magic happens ...

$x=2,\ y=2,\ z=1$ is a max.

Panel 6

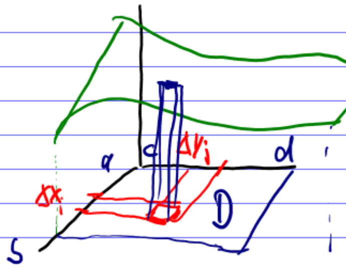
Integration: in $\mathbb{R}$



$$\lim_{n \to \infty} \sum_{i=0}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx$$

Panel 7

Integration in $\mathbb{R}^2$



$$\sum_{i,j=0}^{n} f(x_i,y_j)\,\Delta x_i\,\Delta y_j =$$

$$\iint_D f(x,y)\,dA$$

represents the volume between D and f, if $\underline{f \geq 0}$

Panel 8

Fubini's Theorem (How to integrate in $\mathbb{R}^2$)

If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,d\underline{x} = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

Ex: $\iint_R (x-3y^2)\,dA$, $R=[0,2]\times[1,2]$

$$\int_0^2\int_1^2 (x-3y^2)\,\underline{dy}\,\underline{dx} = \int_0^2\left[\int_1^2 x-3y^2\,dy\right]dx =$$

$$=\int_0^2\left[xy-y^3\Big|_{y=1}^{y=2}\right]dx = \int_0^2 (2x-8)-(x-1)\,dx$$

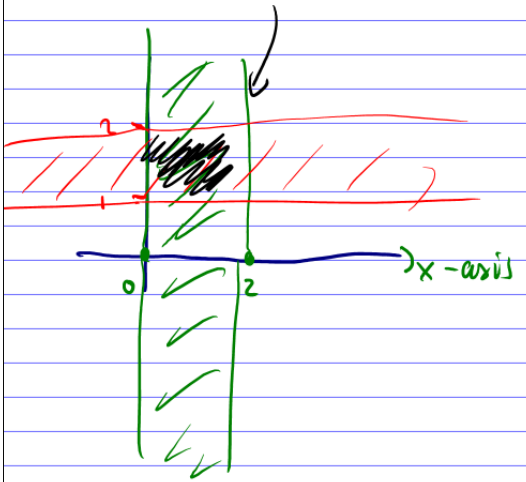
Panel 9

$$\int_0^2 (2x-8)-(x-1)\,dx = \int_0^2 x-7\,dx = \frac{1}{2}x^2-7x\Big|_0^2 = 2-14 = -\underline{\underline{12}}$$

$$\overset{x}{[0,2]} \times \overset{y}{[1,2]}$$

$$\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$$

$$\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \mathbb{R}^3$$



Panel 10

<u>Ex:</u> Find the volume of the solid bounded by $x^2+y^2+z=16$, the planes $x=2$ and $y=2$, and the coordinate planes.

$z = 16-x^2-y^2 = 16-(x^2+y^2)$. z is positive over $[0,2]\times[0,2]$

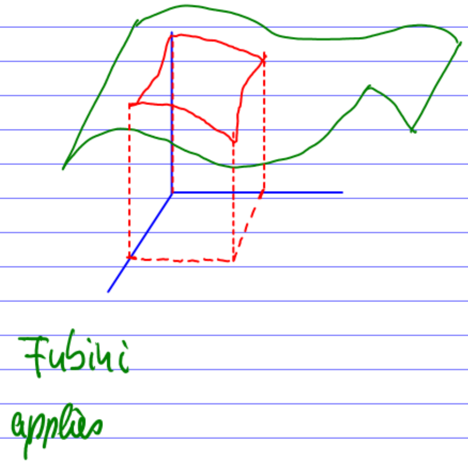
$$\Rightarrow V = \int_0^2\int_0^2 16-x^2-y^2\,dx\,dy =$$

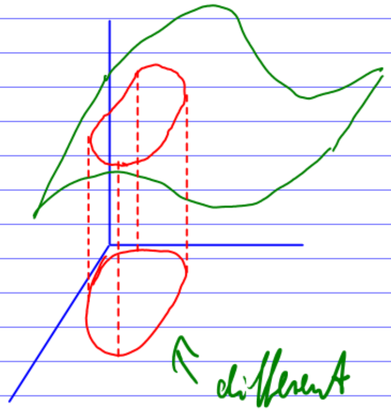
$$= \int_0^2 16x-\frac{1}{3}x^3-xy^2\Big|_{x=0}^{x=2}\,dy =$$

$$= \int_0^2 32-\frac{8}{3}-2y^2\,dy = \frac{84}{3}y-\frac{2}{3}y^3\Big|_0^2 = \frac{164}{3}-\frac{16}{3} = \underline{\underline{\quad}}$$

Panel 11

In $\mathbb{R}$ all we ever did was integrate over intervals $[a,b]$. In $\mathbb{R}^2$ it is different:





Panel 12

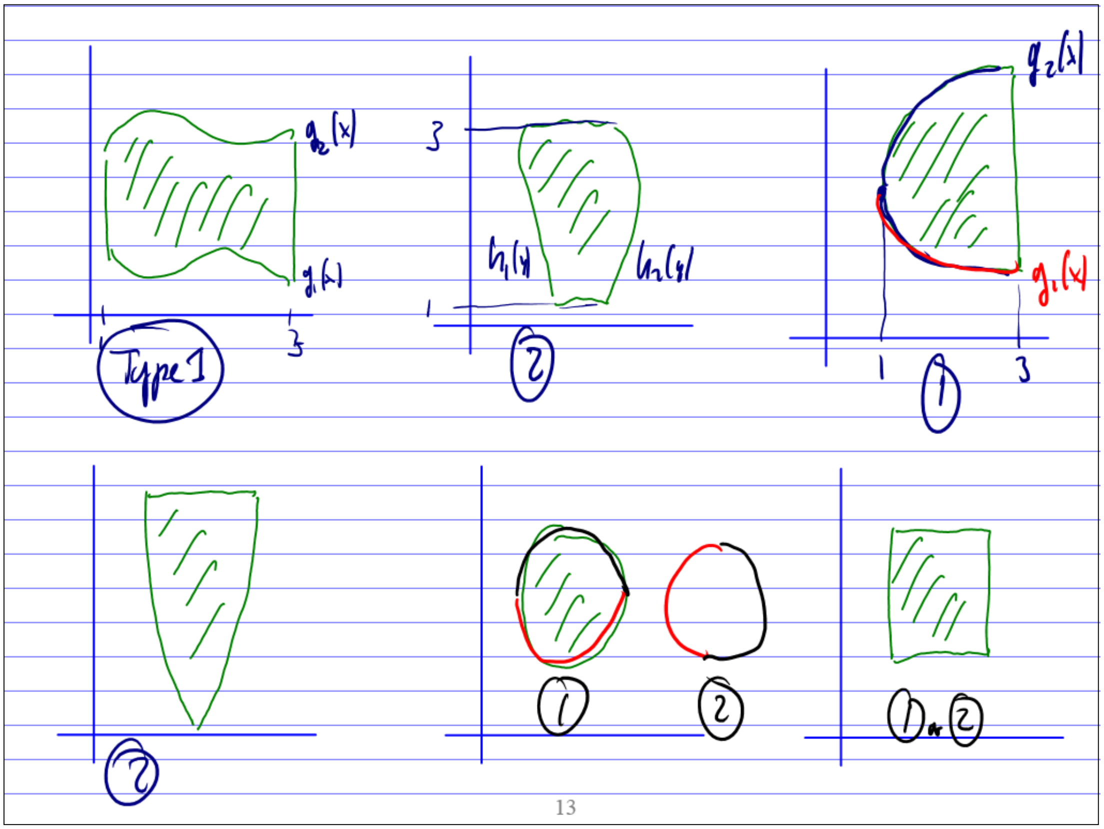
Type I Region: $D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Type 2 Region: $D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

for type-1 regions: $$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

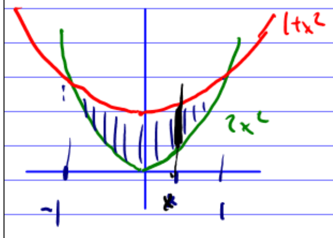
for type-2 regions $$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Panel 13



Panel 14

Ex: Find $\iint_D (x+2y)\,dA$ where D is the region bounded by $y=2x^2$ and $y=1+x^2$



$$\iint_D x+2y\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y\,dy\,dx$$

$$= \int_{-1}^{1} xy+y^2\Big|_{y=2x^2}^{1+x^2}\,dx = \int_{-1}^{1}\left[x(1+x^2)+(1+x^2)^2\right]-\left[x(2x^2)+(2x^2)^2\right]dx$$

$Int(Int(x+2\,y,\,y=2\,x^2\,..1+x^2),\,x=-1\,..1) = int(int(x+2\,y,\,y=2\,x^2\,..1+x^2),\,x=-1\,..1)$

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2\,y)\,dy\,dx = \boxed{\frac{32}{15}}$$

.